TBA or
Numerical stability of a new conformal-traceless 3+1 formulation of the Einstein equation

Deirdre Shoemaker and Pablo Laguna
Center for Gravitational Physics and Geometry, and
Center for Gravitational Wave Physics
Penn State University

May 14 2002

Abstract

The purpose of this talk is to present one potential form of the Einstein equation and the effect that the form of the equations has on both the stability and duration of black-hole evolutions.

Introduction

Present situation: 3+1 evolutions of black hole spacetimes are short-lived.

- BSSN: $t \sim 500M - 600M$ [Alcubierre and Bruegmann, PRD 63, 104006 (2001)]
- Hyperbolic: $t \sim 600M - 1300M$ [Kidder et al, PRD 62, 084032 (2000)]

Do we know why?

There are multiple reasons that are far from being understood:

- gauge and constraint violating modes
- outer boundary conditions
- excision algorithms
- numerical discretization, ...

Focus of this work

The formulation used to recast the Einstein equation as a 3+1 set of evolution equations has a fundamental impact on the stability properties of numerical evolutions.

Fact: The ADM system has failed to produce long-lived simulations, but we do not know exactly why!

Hyperbolicity? It helps, but does not seem to be enough.

Conformal-traceless? Yields improvements, but also not enough.

ADM Formulation

Evolutions eqns:

$$\begin{aligned} \partial_o g_{ij} &= -2\,\alpha\,K_{ij} \\ \partial_o K_{ij} &= -\nabla_i \nabla_j \alpha + \alpha\,R_{ij} + \alpha\,K\,K_{ij} - 2\,\alpha\,K_{ik}\,K^k{}_j \end{aligned}$$

where $\partial_o \equiv \partial_t - \mathcal{L}_\beta$

Constraints:

$$\begin{aligned} R + K^2 - K_{ij}K^{ij} &= 0 \\ \nabla_j K^{ij} - \nabla^i K &= 0 \end{aligned}$$

The Problem(?)

$$\partial_t K_{ij} = \alpha\, K\, K_{ij} + \text{Junk}$$

Consider a t-independent solution. If $\alpha\, K\, K_{ij}$ does not cancel Junk, it is in principle possible to have a solution

$$K \propto e^{\alpha\, K\, t} \qquad \text{!!!!!!}$$

Is there any evidence?

Numerical evolutions of spherically symmetric black hole space-times using an adjusted ADM system [Kelly et al, PRD 64, 084013 (2001)]

Can we do the same in full 3D?

Not with the ADM system.

BSSN

Baumgarte and Shapiro, PRD 59, 024009 (1999); Shibata and Nakamura, PRD 52, 5428 (1995)

Primary variables:

$$
\begin{aligned}
\hat{g}_{ij} &= e^{-4\,\Phi}\, g_{ij} \\
\Phi &= \frac{1}{12} \ln g \\
\hat{A}_{ij} &= e^{-4\,\Phi}\, A_{ij} \\
K &= g^{ij}\, K_{ij} \\
\hat{\Gamma}^i &= \hat{g}^{jk} \hat{\Gamma}^i_{jk}
\end{aligned}
$$

Constraints:

$$
\hat{A}^i{}_i = 0 \quad \hat{g} = 1 \quad \hat{\Gamma}^i = -\partial_j \hat{g}^{ij}
$$

BSSN equations

$$
\begin{aligned}
\partial_o \Phi &= -\frac{1}{6}\alpha\, K \\
\partial_o \hat{g}_{ij} &= -2\,\alpha\,\hat{A}_{ij} \\
\partial_o K &= -\nabla_i \nabla^i \alpha + \alpha\,\hat{A}_{ij}\hat{A}^{ij} + \alpha\, K^2/3 \\
\partial_o \hat{A}_{ij} &= e^{-4\Phi}\left(-\nabla_i \nabla_j \alpha + \alpha\, R_{ij}\right)^{TF} \\
&+ \alpha\, K\, \hat{A}_{ij} - 2\,\alpha\,\hat{A}_{il}\hat{A}^l{}_j \\
\partial_o \widehat{\Gamma}^i &= \hat{g}^{jk}\partial_{jk}\beta^i + \frac{1}{3}\hat{g}^{ij}\partial_{jk}\beta^k - 2\,\hat{A}^{ij}\partial_j \alpha \\
&+ 2\,\alpha \widehat{\Gamma}^i_{jk}\hat{A}^{jk} + 12\,\alpha\hat{A}^{ij}\partial_j \Phi - \frac{4}{3}\alpha\widehat{\nabla}^i K
\end{aligned}
$$

Note: The constraints were used to eliminate R and ∂A

New Conformal-Traceless System

Laguna and Shoemaker gr-qc/0202105

Primary variables:

$$
\begin{aligned}
\hat{g}_{ij} &= e^{-4\,\Phi}\, g_{ij} \\
\Phi &= \frac{1}{12}\ln g \\
\hat{\Gamma}^i &= \hat{g}^{jk}\hat{\Gamma}^i_{jk} \\
\hat{A}^i{}_j &= e^{6\,n\,\Phi}\, A^i{}_j \quad \clubsuit \\
\hat{K} &= e^{6\,n\,\Phi}\, K \quad \clubsuit \\
N &= e^{-6\,n\,\Phi}\, \alpha \quad \clubsuit
\end{aligned}
$$

Note: $A^i{}_j$ has been used before in evolutions of cosmological space-times, but without $\widehat{\Gamma}^i$. [Kurki-Suonio, Laguna and Matzner, PRD, 48, 3611 (1993)]

Evolution Eqns

$$
\begin{aligned}
\partial_o \Phi &= -\frac{1}{6} N\, \hat{K} \\
\partial_o \hat{g}_{ij} &= -2\, N\, \hat{A}_{ij} \\
\partial_o \hat{K} &= -e^{6\,n\,\Phi} \nabla_i \nabla^i \alpha + N\, \hat{A}^i{}_j \hat{A}^j{}_i \\
&+ (1 - 3\,n)\, N\, \hat{K}^2/3 \\
\partial_o \hat{A}^i{}_j &= e^{6\,n\,\Phi} \left(-\nabla^i \nabla_j \alpha + \alpha\, R^i{}_j\right)^{TF} \\
&+ (1 - n)\, N\, \hat{K} \hat{A}^i{}_j \\
\partial_o \widehat{\Gamma}^i &= \hat{g}^{jk} \partial_{jk} \beta^i + \frac{1}{3} \hat{g}^{ij} \partial_{jk} \beta^k - 2\, \hat{A}^{ij} \partial_j N + \ldots
\end{aligned}
$$

Warning:

$\hat{A}^i{}_j$ is not symmetric.

The n-parameter

Recall

$$\partial_t \hat{A}^i{}_j = C\,\hat{A}^i{}_j + \partial \hat{A}\ \text{Terms} + \text{Terms without}\ \hat{A}$$

where

$$C \equiv n\,\partial_k \beta^k + (1-n)\,N\,\hat{K}$$

Thus

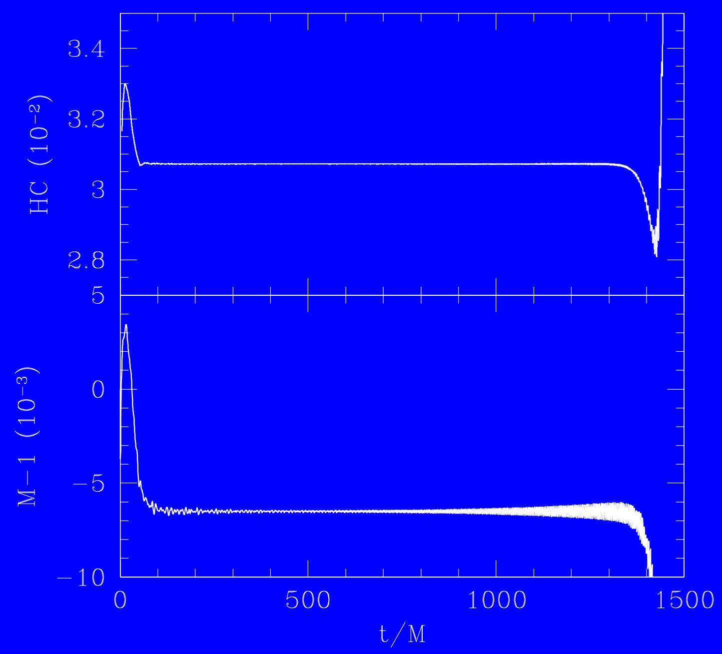
$$n \geq 1 \quad \Rightarrow \quad C < 0 \quad \Rightarrow \quad A^i{}_j \propto e^{C\,t}\ \text{not a problem}$$

Does it work?

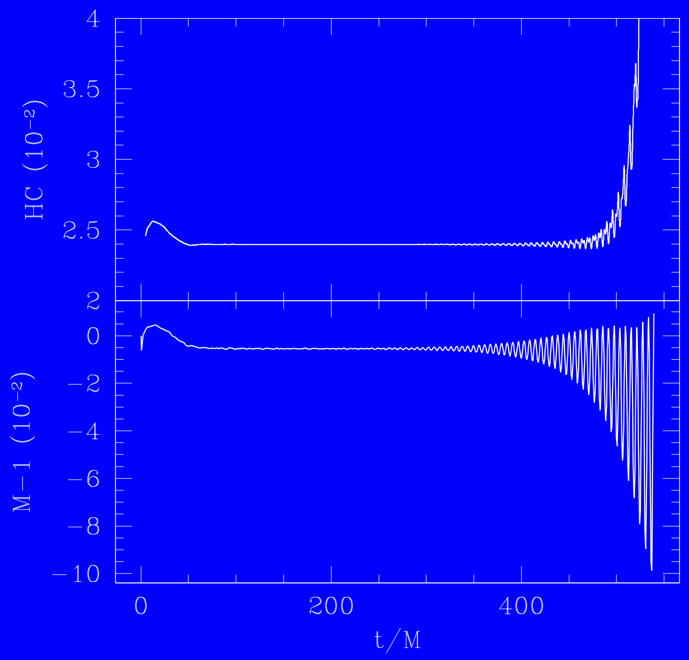
- Infrastructure: CACTUS computational toolkit
- Coordinates: ingoing-Eddington-Finkelstein
- Gauge condition: exact lapse and shift[1]
- Outer boundary: $9M$ with blending
- Grid-spacing: $M/4$
- Excision: cube-shape with soln. extrapolation
- Symmetries: quadrant and bitant

[1]Notice that this choice has only produced evolutions lasting $< 100M$

Hamiltonian and Mass error in quadrant symmetry



Hamiltonian and Mass error in bitant symmetry



Conclusions

Even the algebraic terms in the Einstein equation may have important implications on numerical stability properties of the evolutions.

New Conformal-traceless system:

- Introduced new conformal transformations
- Kept the BSSN conformal connection
- Densitized lapse
- Use $A^i{}_j$ as primary variable
- Test: Single black hole evolutions $t \sim 1400M$

Currently:

- Build our understanding
- Comparison ID and formulations (this meeting in UNAM)
- Decouple densitization of α from the n-parameter
- Implement outer boundary conditions

Thanks: Ashtekar,Fiske,Kelly,Schnetter,Smith