Systematic Understanding of Asymptotical Stability —- via Constraint Propagation Analysis —-

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OUTLINE

Proposals: Adjusted Systems based on ADM/BSSN

Analytic Support: Constraint Propagation eqs.

Much better stability with less efforts?

Refs:

Ashtekar variables	PRL **82** (1999) 263, PRD **60** (1999) 101502, IJMPD **9** (2000) 13, CQG **17** (2000) 4799, CQG **18** (2001) 441
ADM variables	PRD **63** (2001) 120419, CQG **19** (2002) 1027
BSSN variables	gr-qc/0204002

Workshop on Formulations of Einstein's equations for Numerical Relativity @ Mexico, May 2002

Contents:

1 Background of the problem

Numerical Relativity – Necessary for unveiling the nature of strong gravity

– Gravitational Wave from colliding Black Holes, Neutron Stars, Supernovae, ...

– Relativistic Phenomena like Cosmology, Active Galactic Nuclei, ...

– Mathematical feedbacks to Singularity, Exact Solutions, Chaotic behavior, ...

– Labratory of Gravitational theories, Higher dimensional models, ...

Best Einstein formulation for long-term stable and accurate simulation?

Many (too many) trials and errors, not yet a systematical understanding

strategy 0: Arnowitt-Deser-Misner formulation

strategy 1: Shibata-Nakamura's (Baumgarte-Shapiro's) modifications to the standard ADM

strategy 2: Apply a formulation which reveals a hyperbolicity explicitly

strategy 3: Formulate a system which is "asymptotically constrained" against a violation of constraints

The direct use of the standard ADM equations is not recommended.
By adding constraints in RHS, we can kill error growing modes
$\Rightarrow$ How can we understand systematically?

strategy 1 Shibata-Nakamura's (Baumgarte-Shapiro's) modifications to the standard ADM

– define new variables $(\phi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$, instead of the ADM's (γ_{ij}, K_{ij}) where

$$\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}, \qquad \tilde{A}_{ij} \equiv e^{-4\phi}(K_{ij} - (1/3)\gamma_{ij}K), \qquad \tilde{\Gamma}^i \equiv \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk},$$

use momentum constraint in Γ^i-eq., and impose $\det\tilde{\gamma}_{ij} = 1$ during the evolutions.

– The set of evolution equations become

$$\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\phi &= -(1/6)\alpha K, \\
(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij}, \\
(\partial_t - \mathcal{L}_\beta)K &= \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 - \gamma^{ij}(\nabla_i\nabla_j\alpha), \\
(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} &= -e^{-4\phi}(\nabla_i\nabla_j\alpha)^{TF} + e^{-4\phi}\alpha R^{(3)}_{ij} - e^{-4\phi}\alpha(1/3)\gamma_{ij}R^{(3)} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j) \\
\partial_t\tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} - (4/3)\alpha(\partial_j K)\tilde{\gamma}^{ij} + 12\alpha\tilde{A}^{ji}(\partial_j\phi) - 2\alpha\tilde{A}_k{}^j(\partial_j\tilde{\gamma}^{ik}) - 2\alpha\tilde{\Gamma}^k{}_{lj}\tilde{A}^j{}_k\tilde{\gamma}^{il} \\
&\quad -\partial_j\left(\beta^k\partial_k\tilde{\gamma}^{ij} - \tilde{\gamma}^{kj}(\partial_k\beta^i) - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)\right)
\end{aligned}$$

$$\begin{aligned}
R_{ij} &= \partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^m_{ij}\Gamma^k_{mk} - \Gamma^m_{kj}\Gamma^k_{mi} =: \tilde{R}_{ij} + R^\phi_{ij} \\
&\quad R^\phi_{ij} = -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{g}_{ij}\tilde{D}^l\tilde{D}_l\phi + 4(\tilde{D}_i\phi)(\tilde{D}_j\phi) - 4\tilde{g}_{ij}(\tilde{D}^l\phi)(\tilde{D}_l\phi) \\
&\quad \tilde{R}_{ij} = -(1/2)\tilde{g}^{lm}\partial_{lm}\tilde{g}_{ij} + \tilde{g}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + 2\tilde{g}^{lm}\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{g}lm\tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}
\end{aligned}$$

– No explicit explanations why this formulation works better.
Potsdam group (2000): the replacement by momentum constraint is essential.

strategy 2 Apply a formulation which reveals a hyperbolicity explicitly.

For a first order partial differential equations on a vector u,

$$\partial_t \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} & A & \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{characteristic part}} + \underbrace{B \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{lower order part}} \tag{1}$$

	if the eigenvalues of A are
weakly hyperbolic	all real.
strongly hyperbolic	all real and $\exists$ a complete set of eigenvalues.
symmetric hyperbolic	if A is real and symmetric (Hermitian).

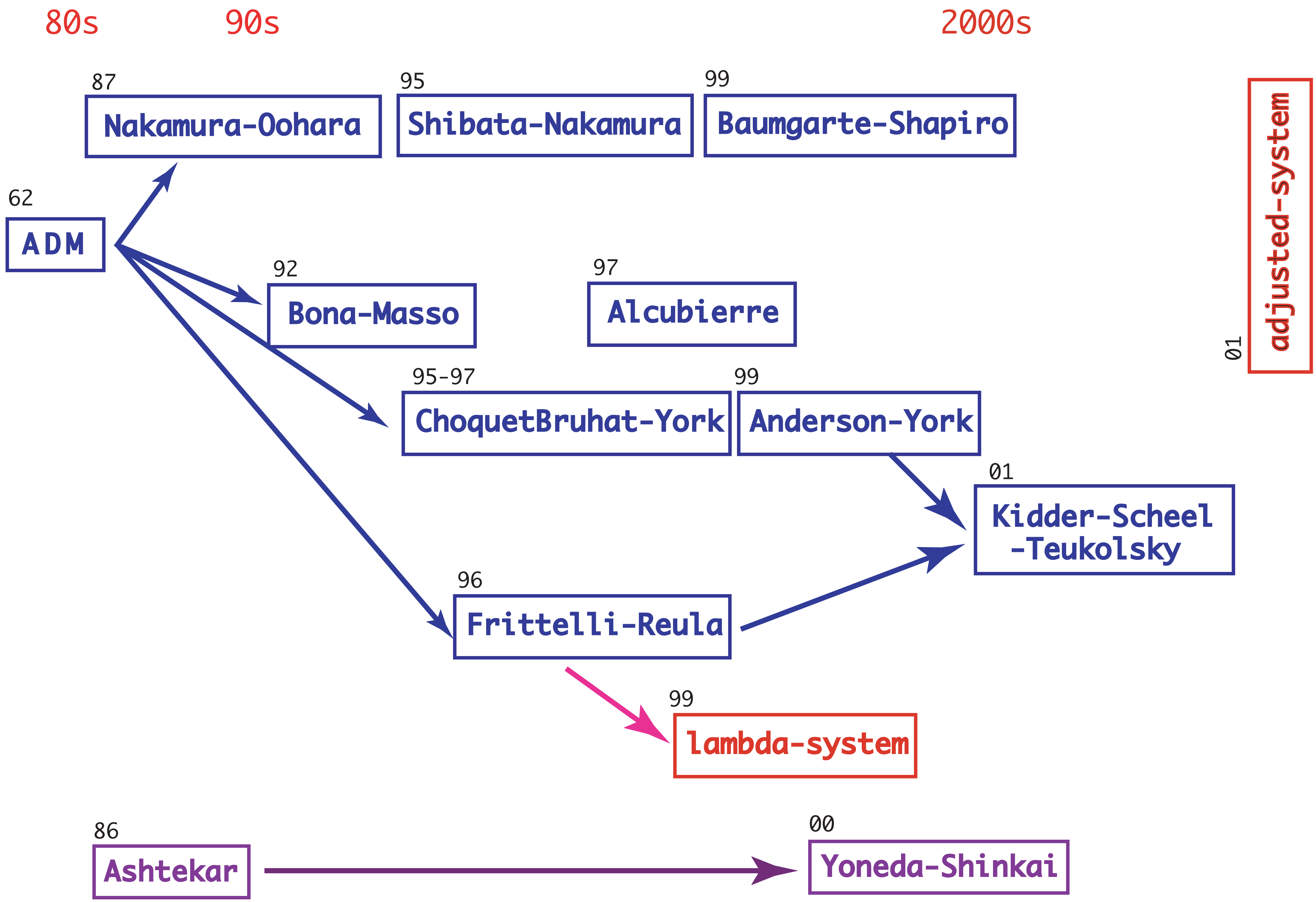
Expectations

– Wellposed behaviour

symmetric hyperbolic system $\Longrightarrow$ WELL-POSED , $||u(t)|| \leq e^{\kappa t}||u(0)||$

– Better boundary treatments $\Longleftarrow \exists$ characteristic field.

– known numerical techniques in Newtonian hydrodynamics.



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Shibata-Nakamura
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Baumgarte-Shapiro
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adjusted-system
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Kidder-Scheel
-Teukolsky
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Frittelli-Reula
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lambda-system
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Ashtekar
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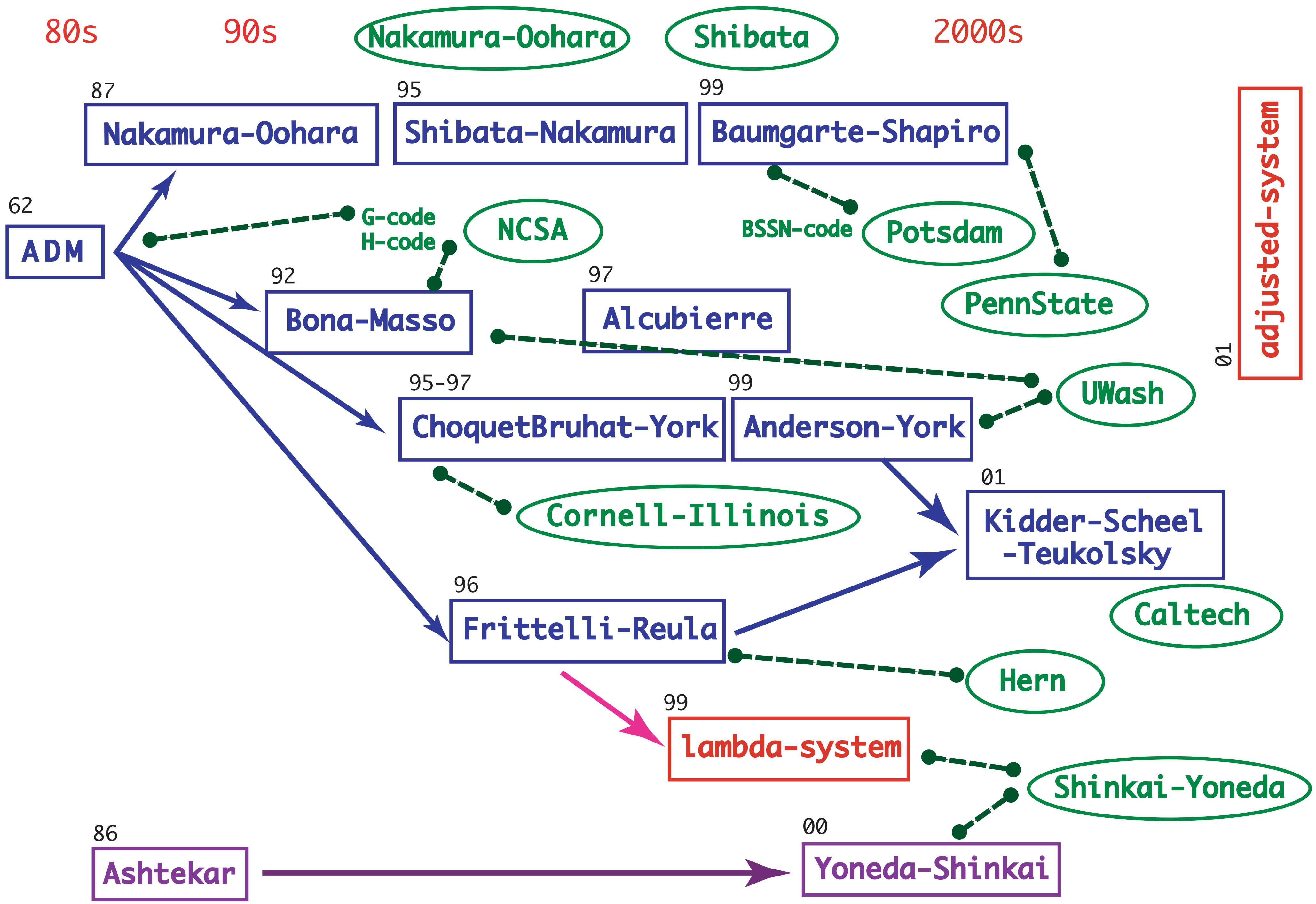
Yoneda-Shinkai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strategy 2 Apply a formulation which reveals a hyperbolicity explicitly. (cont.)

weakly hyperbolic $\ni$ strongly hyperbolic $\ni$ symmetric hyperbolic systems,

Are they actually helpful? Which level of hyperbolicity is necessary?

Using Ashtekar's variables between we found that [HS-Yoneda, CQG17(2000)4799]

(1) the three levels of hyperbolicity can be obtained by adding constraint terms and/or imposing gauge conditions

(2) there is no drastic difference in the accuracy of numerical evolutions in these three levels (comparison of nonlinear wave propagation in a plane symmetric spacetime)

(3) the symmetric hyperbolic system is not always the best for reducing numerical errors

Note that IBVP (Initial Boundary Value problem) requires "symmetric hyperbolicity" to be treated with.

Kidder-Scheel-Teukolsky hyperbolic formulation (Anderson-York + Frittelli-Reula)

Phys. Rev. D. 64 (2001) 064017

- Construct a First-order form using variables $(K_{ij}, g_{ij}, d_{kij})$ where $d_{kij} \equiv \partial_k g_{ij}$
  Constraints are $(\mathcal{H}, \mathcal{M}_i, \mathcal{C}_{kij}, \mathcal{C}_{klij})$ where $\mathcal{C}_{kij} \equiv d_{kij} - \partial_k g_{ij}$, and $\mathcal{C}_{klij} \equiv \partial_{[k} d_{l]ij}$
- Densitize the lapse, $Q = \log(N g^{-\sigma})$
- Adjust equations with constraints

$$\begin{aligned}
\hat{\partial}_0 g_{ij} &= -2NK_{ij} \\
\hat{\partial}_0 K_{ij} &= (\cdots) + \gamma N g_{ij}\mathcal{H} + \zeta N g^{ab}\mathcal{C}_{a(ij)b} \\
\hat{\partial}_0 d_{kij} &= (\cdots) + \eta N g_{k(i}\mathcal{M}_{j)} + \chi N g_{ij}\mathcal{M}_k
\end{aligned}$$

- Re-deining the variables $(P_{ij}, g_{ij}, M_{kij})$

$$\begin{aligned}
P_{ij} &\equiv K_{ij} + \hat{z} g_{ij} K, \\
M_{kij} &\equiv (1/2)[\hat{k} d_{kij} + \hat{e} d_{(ij)k} + g_{ij}(\hat{a} d_k + \hat{b} b_k) + g_{k(i}(\hat{c} d_{j)} + \hat{d} b_{j)})], \quad d_k = g^{ab} d_{kab}, b_k = g^{ab} d_{abk}
\end{aligned}$$

The redefinition parameters

– do not change the eigenvalues of evolution eqs.

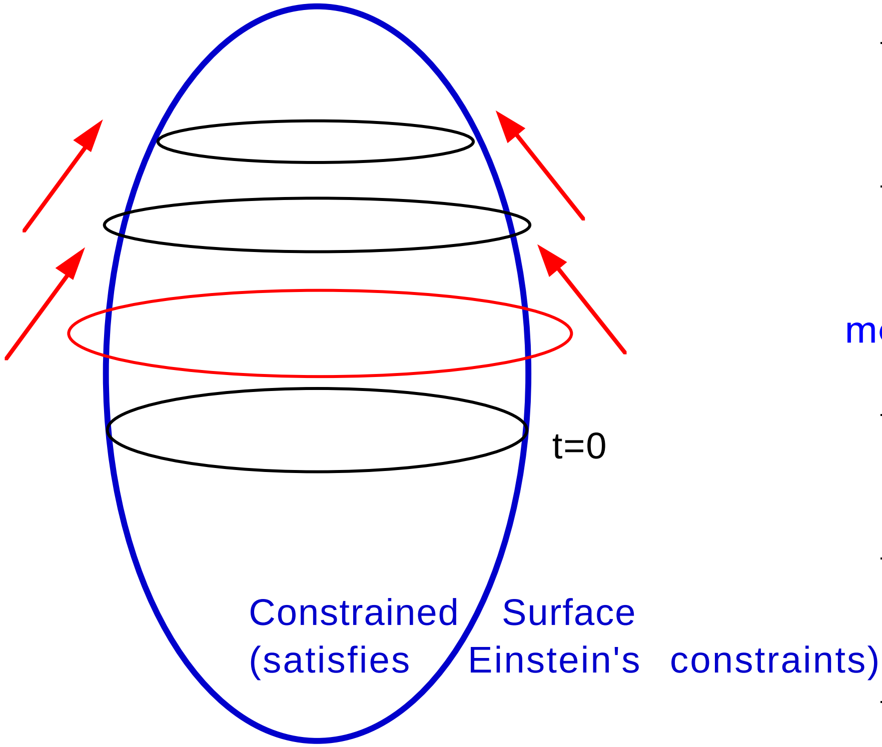
– do not effect on the principal part of the constraint evolution eqs.

– do affect the eigenvectors of evolution system.

– do affect nonlinear terms of evolution eqs/constraint evolution eqs.

strategy 3 Formulate a system which is "asymptotically constrained" against a violation of constraints

"Asymptotically Constrained System"– Constraint Surface as an Attractor



method 1: λ-system (Brodbeck et al, 2000)

– Add aritificial force to reduce the violation of constraints

– To be guaranteed if we apply the idea to a symmetric hyperbolic system.

method 2: Adjusted system (Yoneda HS, 2000, 2001)

– We can control the violation of constraints by adjusting constraints to EoM.

– Eigenvalue analysis of constraint propagation equations may prodict the violation of error.

– This idea is applicable even if the system is not symmetric hyperbolic. $\Rightarrow$ for the ADM formulation, too!!

Idea of λ-system

Brodbeck, Frittelli, Hübner and Reula, JMP40(99)909

We expect a system that is robust for controlling the violation of constraints

Recipe

1. prepare symmetric hyperbolic evolution system $\quad \partial_t u = J u' + K$
2. confirm the evolution of constraint behaves good $\quad \partial_t C = D C' + E C$
3. introduce λ as an indicator of violating constraint $\quad \partial_t \lambda = \alpha C - \beta \lambda$
   initially $\lambda = 0$, obey dissipative eqs. of motion $\quad (\alpha \neq 0, \beta > 0)$
4. take a set of (u, λ) as a dynamical system $\quad \partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} \simeq \begin{pmatrix} A & 0 \\ F & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix}'$
5. modify evolution equation so as to
   form a symmetric hyperbolic system $\quad \partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} A & \bar{F} \\ F & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix}'$

Remarks

- BFHR used a sym. hyp. formulation by Frittelli-Reula [PRL76(96)4667]
- Version for the Ashtekar formulation by HS-Yoneda [PRD60(99)101502]
  for controlling the constraints or reality conditions or both.
- succeeded in evolution of GW in planar spacetime using Ashtekar vars. [CQG18(2001)441]

Idea of "Adjusted system" and Our Conjecture

CQG18 (2001) 441, PRD 63 (2001) 120419, CQG 19 (2002) 1027

General Procedure

1. prepare a set of evolution eqs. $\partial_t u^a = f(u^a, \partial_b u^a, \cdots)$
2. add constraints in RHS $\partial_t u^a = f(u^a, \partial_b u^a, \cdots) + F(C^a, \partial_b C^a, \cdots)$
3. choose appropriate $F(C^a, \partial_b C^a, \cdots)$ to make the system stable evolution

How to specify $F(C^a, \partial_b C^a, \cdots)$?

4. prepare constraint propagation eqs. $\partial_t C^a = g(C^a, \partial_b C^a, \cdots)$
5. and its adjusted version $\partial_t C^a = g(C^a, \partial_b C^a, \cdots) + G(C^a, \partial_b C^a, \cdots)$
6. Fourier transform and evaluate eigenvalues $\partial_t \hat{C}^k = A(\hat{C}^a)\, \hat{C}^k$

Conjecture: Evaluate eigenvalues of (Fourier-transformed) constraint propagation eqs.
If their (1) real part is non-positive, or (2) imaginary part is non-zero, then the system is more stable.

3 Adjusted ADM systems

We adjust the standard ADM system using constraints as:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \quad (1)$$

$$+P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \quad (2)$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k)K_{kj} + (\nabla_j \beta^k)K_{ki} + \beta^k \nabla_k K_{ij}, \quad (3)$$

$$+R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \quad (4)$$

with constraint equations

$$\mathcal{H} := R^{(3)} + K^2 - K_{ij}K^{ij}, \quad (5)$$

$$\mathcal{M}_i := \nabla_j K^j{}_i - \nabla_i K. \quad (6)$$

We can write the adjusted constraint propagation equations as

$$\partial_t \mathcal{H} = \text{(original terms)} + H_1^{mn}[(2)] + H_2^{imn}\partial_i[(2)] + H_3^{ijmn}\partial_i\partial_j[(2)] + H_4^{mn}[(4)], \quad (7)$$

$$\partial_t \mathcal{M}_i = \text{(original terms)} + M_{1i}{}^{mn}[(2)] + M_{2i}{}^{jmn}\partial_j[(2)] + M_{3i}{}^{mn}[(4)] + M_{4i}{}^{jmn}\partial_j[(4)]. \quad (8)$$

The constraint propagation equations of the original ADM equation:

- Expression using $\mathcal{H}$ and $\mathcal{M}_i$ (1)

$$\begin{aligned}
\partial_t \mathcal{H} &= \beta^j(\partial_j \mathcal{H}) + 2\alpha K \mathcal{H} - 2\alpha\gamma^{ij}(\partial_i \mathcal{M}_j) + \alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i, \\
\partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^j(\partial_j \mathcal{M}_i) + \alpha K \mathcal{M}_i - \beta^k \gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j.
\end{aligned}$$

- Expression using $\mathcal{H}$ and $\mathcal{M}_i$ (2)

$$\begin{aligned}
\partial_t \mathcal{H} &= \beta^l \partial_l \mathcal{H} + 2\alpha K \mathcal{H} - 2\alpha\gamma^{-1/2}\partial_l(\sqrt{\gamma}\mathcal{M}^l) - 4(\partial_l \alpha)\mathcal{M}^l \\
&= \beta^l \nabla_l \mathcal{H} + 2\alpha K \mathcal{H} - 2\alpha(\nabla_l \mathcal{M}^l) - 4(\nabla_l \alpha)\mathcal{M}^l, \\
\partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^l \nabla_l \mathcal{M}_i + \alpha K \mathcal{M}_i + (\nabla_i \beta_l)\mathcal{M}^l \\
&= -(1/2)\alpha(\nabla_i \mathcal{H}) - (\nabla_i \alpha)\mathcal{H} + \beta^l \nabla_l \mathcal{M}_i + \alpha K \mathcal{M}_i + (\nabla_i \beta_l)\mathcal{M}^l,
\end{aligned}$$

- Expression using $\mathcal{H}$ and $\mathcal{M}_i$ (3): by using Lie derivatives along αn^μ,

$$\begin{aligned}
\pounds_{\alpha n^\mu} \mathcal{H} &= +2\alpha K \mathcal{H} - 2\alpha\gamma^{-1/2}\partial_l(\sqrt{\gamma}\mathcal{M}^l) - 4(\partial_l \alpha)\mathcal{M}^l, \\
\pounds_{\alpha n^\mu} \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \alpha K \mathcal{M}_i.
\end{aligned}$$

- Expression using γ_{ij} and K_{ij}

$$\begin{aligned}
\partial_t \mathcal{H} &= H_1^{mn}(\partial_t \gamma_{mn}) + H_2^{imn}\partial_i(\partial_t \gamma_{mn}) + H_3^{ijmn}\partial_i \partial_j(\partial_t \gamma_{mn}) + H_4^{mn}(\partial_t K_{mn}), \\
\partial_t \mathcal{M}_i &= M_{1i}{}^{mn}(\partial_t \gamma_{mn}) + M_{2i}{}^{jmn}\partial_j(\partial_t \gamma_{mn}) + M_{3i}{}^{mn}(\partial_t K_{mn}) + M_{4i}{}^{jmn}\partial_j(\partial_t K_{mn}),
\end{aligned}$$

where

$$
\begin{aligned}
H_1^{mn} &:= -2R^{(3)mn} - \Gamma^p_{kj}\Gamma^k_{pi}\gamma^{mi}\gamma^{nj} + \Gamma^m\Gamma^n \\
&\quad +\gamma^{ij}\gamma^{np}(\partial_i\gamma^{mk})(\partial_j\gamma_{kp}) - \gamma^{mp}\gamma^{ni}(\partial_i\gamma^{kj})(\partial_j\gamma_{kp}) - 2KK^{mn} + 2K^n{}_jK^{mj}, \\
H_2^{imn} &:= -2\gamma^{mi}\Gamma^n - (3/2)\gamma^{ij}(\partial_j\gamma^{mn}) + \gamma^{mj}(\partial_j\gamma^{in}) + \gamma^{mn}\Gamma^i, \\
H_3^{ijmn} &:= -\gamma^{ij}\gamma^{mn} + \gamma^{in}\gamma^{mj}, \\
H_4^{mn} &:= 2(K\gamma^{mn} - K^{mn}), \\
M_{1i}{}^{mn} &:= \gamma^{nj}(\partial_i K^m{}_j) - \gamma^{mj}(\partial_j K^n{}_i) + (1/2)(\partial_j\gamma^{mn})K^j{}_i + \Gamma^n K^m{}_i, \\
M_{2i}{}^{jmn} &:= -\gamma^{mj}K^n{}_i + (1/2)\gamma^{mn}K^j{}_i + (1/2)K^{mn}\delta^j_i, \\
M_{3i}{}^{mn} &:= -\delta^n_i\Gamma^m - (1/2)(\partial_i\gamma^{mn}), \\
M_{4i}{}^{jmn} &:= \gamma^{mj}\delta^n_i - \gamma^{mn}\delta^j_i,
\end{aligned}
$$

where we expressed $\Gamma^m = \Gamma^m_{ij}\gamma^{ij}$.

Original ADM The original construction by ADM uses the pair of (h_{ij}, π^{ij}).

$$\mathcal{L} = \sqrt{-g}R = \sqrt{h}N[{}^{(3)}R - K^2 + K_{ij}K^{ij}], \quad \text{where } K_{ij} = \frac{1}{2}\pounds_n h_{ij}$$

$$\text{then} \quad \pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \sqrt{h}(K^{ij} - Kh^{ij}),$$

The Hamiltonian density gives us constraints and evolution eqs.

$$\mathcal{H} = \pi^{ij}\dot{h}_{ij} - \mathcal{L} = \sqrt{h}\left\{N\mathcal{H}(h,\pi) - 2N_j\mathcal{M}^j(h,\pi) + 2D_i(h^{-1/2}N_j\pi^{ij})\right\},$$

$$\begin{cases} \partial_t h_{ij} = \dfrac{\delta \mathcal{H}}{\delta \pi^{ij}} = 2\dfrac{N}{\sqrt{h}}(\pi_{ij} - \dfrac{1}{2}h_{ij}\pi) + 2D_{(i}N_{j)}, \\ \partial_t \pi^{ij} = -\dfrac{\delta \mathcal{H}}{\delta h_{ij}} = -\sqrt{h}N({}^{(3)}R^{ij} - \dfrac{1}{2}{}^{(3)}Rh^{ij}) + \dfrac{1}{2}\dfrac{N}{\sqrt{h}}h^{ij}(\pi_{mn}\pi^{mn} - \dfrac{1}{2}\pi^2) - 2\dfrac{N}{\sqrt{h}}(\pi^{in}\pi_n{}^j - \dfrac{1}{2}\pi\pi^{ij}) \\ \qquad + \sqrt{h}(D^iD^jN - h^{ij}D^mD_mN) + \sqrt{h}D_m(h^{-1/2}N^m\pi^{ij}) - 2\pi^{m(i}D_mN^{j)} \end{cases}$$

Standard ADM (by York) NRists refer ADM as the one by York with a pair of (h_{ij}, K_{ij}).

$$\begin{cases} \partial_t h_{ij} = -2NK_{ij} + D_jN_i + D_iN_j, \\ \partial_t K_{ij} = N({}^{(3)}R_{ij} + KK_{ij}) - 2NK_{il}K^l{}_j - D_iD_jN + (D_jN^m)K_{mi} + (D_iN^m)K_{mj} + N^mD_mK_{ij} \end{cases}$$

In the process of converting, $\mathcal{H}$ was used, i.e. the standard ADM has already adjusted.

3 Constraint propagation of ADM systems

3.1 Original ADM vs Standard ADM

Try the adjustment $\underline{R_{ij} = \kappa_1 \alpha \gamma_{ij}}$ and other multiplier zero, where $\kappa_1 = \begin{cases} 0 & \text{the standard ADM} \\ -1/4 & \text{the original ADM} \end{cases}$

- The constraint propagation eqs keep the first-order form (cf Frittelli, PRD55(97)5992):

$$\partial_t \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_i \end{pmatrix} \simeq \begin{pmatrix} \beta^l & -2\alpha\gamma^{jl} \\ -(1/2)\alpha\delta^l_i + R^l{}_i - \delta^l_i R & \beta^l \delta^j_i \end{pmatrix} \partial_l \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_j \end{pmatrix}. \tag{5}$$

  The eigenvalues of the characteristic matrix:

$$\lambda^l = (\beta^l, \beta^l, \beta^l \pm \sqrt{\alpha^2 \gamma^{ll}(1+4\kappa_1)})$$

  The hyperbolicity of (5): $\begin{cases} \text{symmetric hyperbolic} & \text{when } \kappa_1 = 3/2 \\ \text{strongly hyperbolic} & \text{when } \alpha^2\gamma^{ll}(1+4\kappa_1) > 0 \\ \text{weakly hyperbolic} & \text{when } \alpha^2\gamma^{ll}(1+4\kappa_1) \geq 0 \end{cases}$

- On the Minkowskii background metric, the linear order terms of the Fourier-transformed constraint propagation equations gives the eigenvalues

$$\Lambda^l = (0, 0, \pm\sqrt{-k^2(1+4\kappa_1)}).$$

  That is, $\begin{cases} \text{(two 0s, two pure imaginary)} & \text{for the standard ADM} \\ \text{(four 0s)} & \text{for the original ADM} \end{cases}$ BETTER STABILITY

3.2 Detweiler's system

3.2.1 Detweiler's system and its constraint amplification

Detweiler's modification to ADM [PRD35(87)1095] can be realized in our notation as:

$$\begin{aligned} P_{ij} &= -L\alpha^3\gamma_{ij}, \\ R_{ij} &= L\alpha^3(K_{ij} - (1/3)K\gamma_{ij}), \\ S^k_{ij} &= L\alpha^2[3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l\alpha)\gamma_{ij}\gamma^{kl}], \\ s^{kl}_{ij} &= L\alpha^3[2\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}], \end{aligned}$$

and else zero, where L is a constant.

- This adjustment does not make constraint propagation equation in the first order form, so that we can not discuss the hyperbolicity nor the characteristic speed of the constraints.
- For the Minkowskii background spacetime, the adjusted constraint propagation equations with above choice of multiplier become

$$\begin{aligned} \partial_t {}^{(1)}\mathcal{H} &= -2(\partial_j {}^{(1)}\mathcal{M}_j) + 4L(\partial_j\partial_j {}^{(1)}\mathcal{H}), \\ \partial_t {}^{(1)}\mathcal{M}_i &= -(1/2)(\partial_i {}^{(1)}\mathcal{H}) + (L/2)(\partial_k\partial_k {}^{(1)}\mathcal{M}_i) + (L/6)(\partial_i\partial_k {}^{(1)}\mathcal{M}_k). \end{aligned}$$

The eigenvalues of their Fourier expression are

$$\Lambda^l = (-(L/2)k^2(\text{multiplicity } 2), -(7L/3)k^2 \pm (1/3)\sqrt{k^2(-9 + 25L^2k^2)}.)$$

This indicates negative real eigenvalues if we chose small positive L.

3.2.2 Numerical demonstration

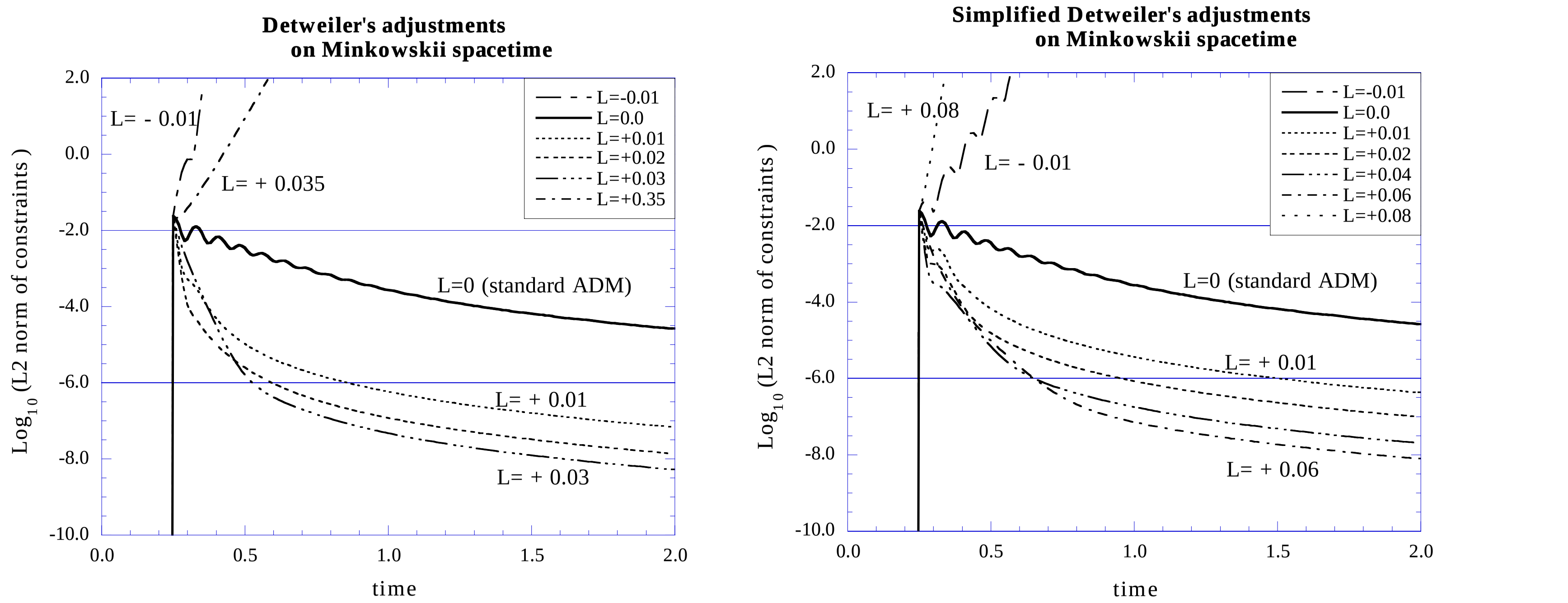


Figure 1: We confirmed numerically, using Minkowskii perturbation, that Detweiler's system presents better accuracy than the standard ADM, but only form small positive L.

3.2.3 Differences with Detweiler's requirement

Detweiler calculated the L2 norm of the constraints, C_α, over the 3-hypersurface and imposed its negative definiteness of its evolution,

$$\text{Detweiler's criteria} \;\Leftrightarrow\; \partial_t \int \sum_\alpha C_\alpha^2 \, dV < 0,$$

This is rewritten by supposing the constraint propagation to be $\partial_t \hat{C}_\alpha = A_\alpha{}^\beta \hat{C}_\beta$ in the Fourier components,

$$\Leftrightarrow \quad \partial_t \int \sum_\alpha \hat{C}_\alpha \bar{\hat{C}}_\alpha \, dV = \int \sum_\alpha A_\alpha{}^\beta \hat{C}_\beta \bar{\hat{C}}_\alpha + \hat{C}_\alpha \bar{A}_\alpha{}^\beta \bar{\hat{C}}_\beta \, dV < 0, \ \forall \text{ non zero } \hat{C}_\alpha$$

$$\Leftrightarrow \quad \underline{\text{eigenvalues of } (A + A^\dagger) \text{ are all negative for } \forall k}.$$

On the other hand, our criteria is that the eigenvalues of A are all negative. Therefore,

Our criteria $\ni$ Detweiler's criteria

We remark that Detweiler's truncations on higher order terms in C-norm corresponds our perturbative analysis, both based on the idea that the deviations from constraint surface (the errors expressed non-zero constraint value) are initially small.

4 Constraint propagations in spherically symmetric spacetime

4.1 The procedure

The discussion becomes clear if we expand the constraint $C_\mu := (\mathcal{H}, \mathcal{M}_i)^T$ using vector harmonics.

$$C_\mu = \sum_{l,m} \left(A^{lm}(t,r) a_{lm}(\theta,\varphi) + B^{lm} b_{lm} + C^{lm} c_{lm} + D^{lm} d_{lm} \right), \tag{1}$$

where we choose the basis of the vector harmonics as

$$a_{lm} = \begin{pmatrix} Y_{lm} \\ 0 \\ 0 \\ 0 \end{pmatrix}, b_{lm} = \begin{pmatrix} 0 \\ Y_{lm} \\ 0 \\ 0 \end{pmatrix}, c_{lm} = \frac{r}{\sqrt{l(l+1)}} \begin{pmatrix} 0 \\ 0 \\ \partial_\theta Y_{lm} \\ \partial_\varphi Y_{lm} \end{pmatrix}, d_{lm} = \frac{r}{\sqrt{l(l+1)}} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\sin\theta}\partial_\varphi Y_{lm} \\ \sin\theta\, \partial_\theta Y_{lm} \end{pmatrix}.$$

The basis are normalized so that they satisfy

$$\langle C_\mu, C_\nu \rangle = \int_0^{2\pi} d\varphi \int_0^\pi C_\mu^* C_\rho \, \eta^{\nu\rho} \sin\theta d\theta,$$

where $\eta^{\nu\rho}$ is Minkowskii metric and the asterisk denotes the complex conjugate. Therefore

$$A^{lm} = \langle a^{lm}_{(\nu)}, C_\nu \rangle, \quad \partial_t A^{lm} = \langle a^{lm}_{(\nu)}, \partial_t C_\nu \rangle, \quad \text{etc.}$$

We also express these evolution equations using the Fourier expansion on the radial coordinate,

$$A^{lm} = \sum_k \hat{A}^{lm}_{(k)}(t)\, e^{ikr} \quad \text{etc.} \tag{2}$$

So that we will be able to obtain the RHS of the evolution equations for $(\hat{A}^{lm}_{(k)}(t), \cdots, \hat{D}^{lm}_{(k)}(t))^T$ in a homogeneous form.

4.2 Constraint propagations in Schwarzschild spacetime

1. the standard Schwarzschild coordinate

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2, \qquad \text{(the standard expression)}$$

2. the isotropic coordinate, which is given by, $r = (1 + M/2r_{iso})^2 r_{iso}$:

$$ds^2 = -(\frac{1 - M/2r_{iso}}{1 + M/2r_{iso}})^2 dt^2 + (1 + \frac{M}{2r_{iso}})^4 [dr_{iso}^2 + r_{iso}^2 d\Omega^2], \qquad \text{(the isotropic expression)}$$

3. the ingoing Eddington-Finkelstein (iEF) coordinate, by $t_{iEF} = t + 2M \log(r - 2M)$:

$$ds^2 = -(1 - \frac{2M}{r})dt_{iEF}^2 + \frac{4M}{r} dt_{iEF} dr + (1 + \frac{2M}{r})dr^2 + r^2 d\Omega^2 \qquad \text{(the iEF expression)}$$

4. the Painlevé-Gullstrand (PG) coordinates,

$$ds^2 = -\left(1 - \frac{2\,M}{r}\right) dt_{PG}^2 + 2\sqrt{\frac{2\,M}{r}} dt_{PG}\, dr + dr^2 + r^2 d\Omega^2 \,, \qquad \text{(the PG expression)}$$

which is given by $t_{PG} = t + \sqrt{8Mr} - 2M \log\{(\sqrt{r/2M} + 1)/(\sqrt{r/2M} - 1)\}$

Example 1: standard ADM vs original ADM (in Schwarzschild coordinate)

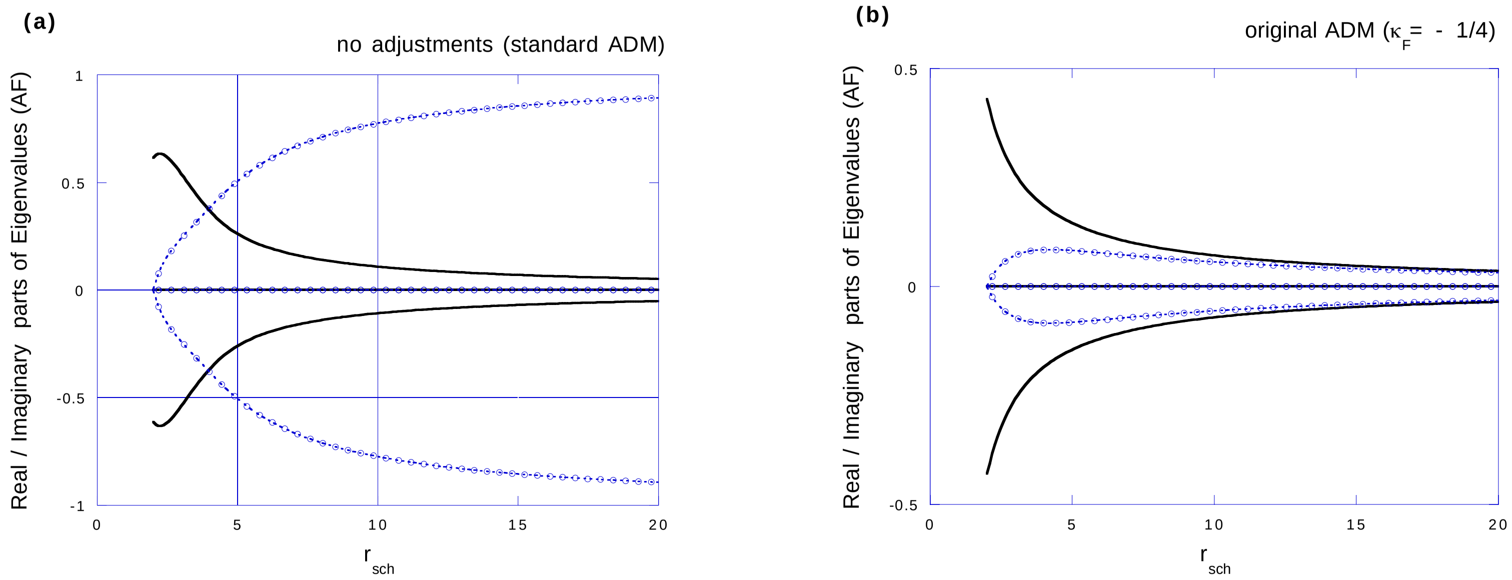


Figure 1: Amplification factors (AFs, eigenvalues of homogenized constraint propagation equations) are shown for the standard Schwarzschild coordinate, with (a) no adjustments, i.e., standard ADM, (b) original ADM ($\kappa_F = -1/4$). The solid lines and the dotted lines with circles are real parts and imaginary parts, respectively. They are four lines each, but actually the two eigenvalues are zero for all cases. Plotting range is $2 < r \leq 20$ using Schwarzschild radial coordinate. We set $k = 1, l = 2$, and $m = 2$ throughout the article.

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i,$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} + \kappa_F \alpha \gamma_{ij} \mathcal{H},$$

Example 2: Detweiler-type adjusted (in Schwarzschild coord.)

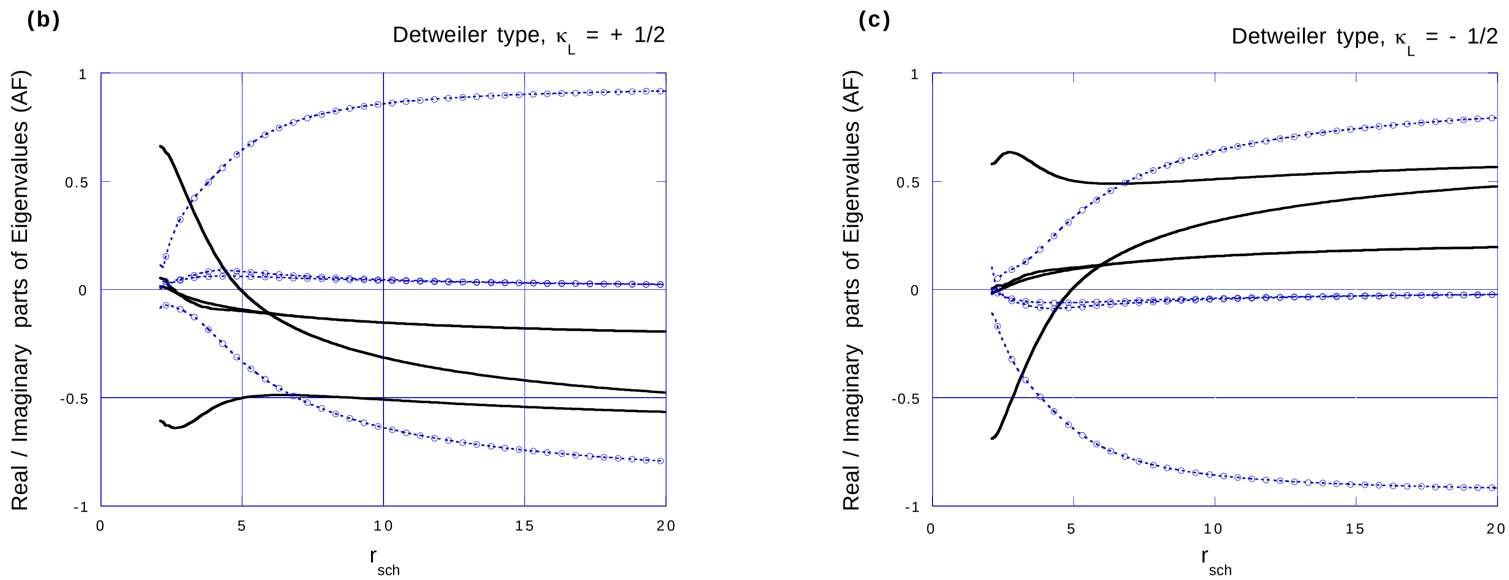


Figure 2: Amplification factors of the standard Schwarzschild coordinate, with Detweiler type adjustments. Multipliers used in the plot are (b) $\kappa_L = +1/2$, and (c) $\kappa_L = -1/2$.

$$\partial_t \gamma_{ij} = \text{(original terms)} + P_{ij}\mathcal{H},$$

$$\partial_t K_{ij} = \text{(original terms)} + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l),$$

where $P_{ij} = -\kappa_L \alpha^3 \gamma_{ij}, \quad R_{ij} = \kappa_L \alpha^3 (K_{ij} - (1/3)K\gamma_{ij}),$

$$S^k{}_{ij} = \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}], \quad s^{kl}{}_{ij} = \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}],$$

Example 3: standard ADM (in isotropic/iEF coord.)

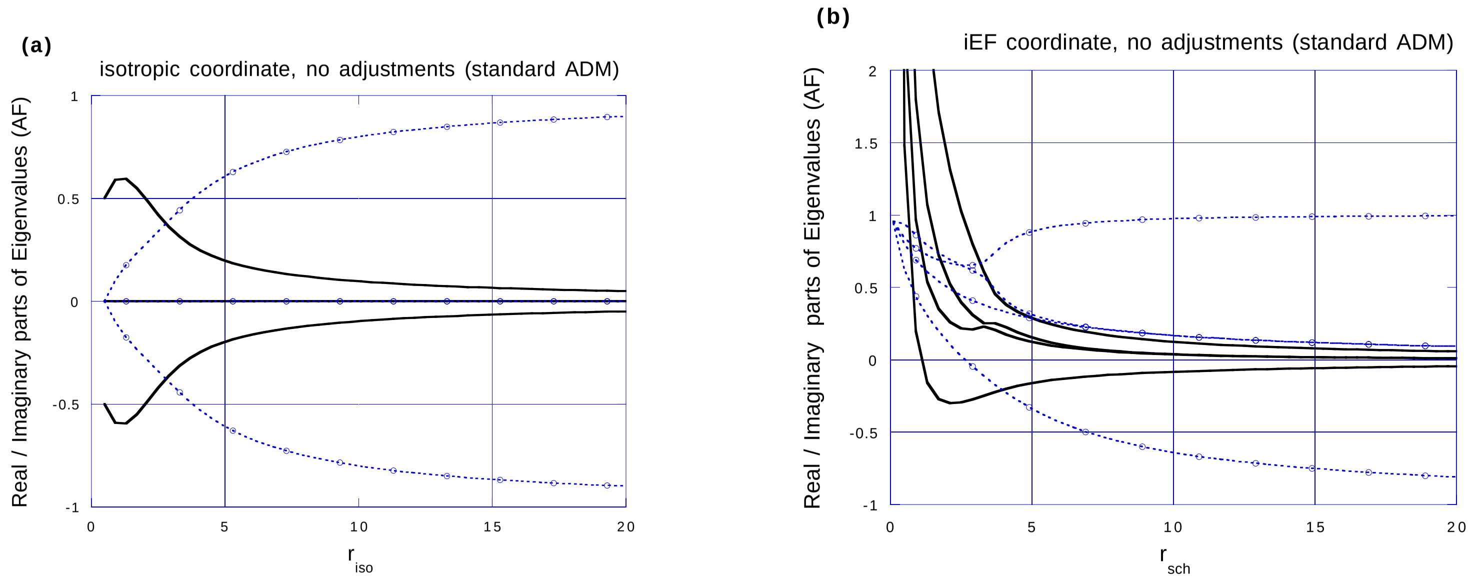


Figure 3: Comparison of amplification factors between different coordinate expressions for the standard ADM formulation (i.e. no adjustments). Fig. (a) is for the isotropic coordinate (1), and the plotting range is $1/2 \leq r_{iso}$. Fig. (b) is for the iEF coordinate (1) and we plot lines on the $t = 0$ slice for each expression. The solid four lines and the dotted four lines with circles are real parts and imaginary parts, respectively.

Example 4: Detweiler-type adjusted (in iEF/PG coord.)

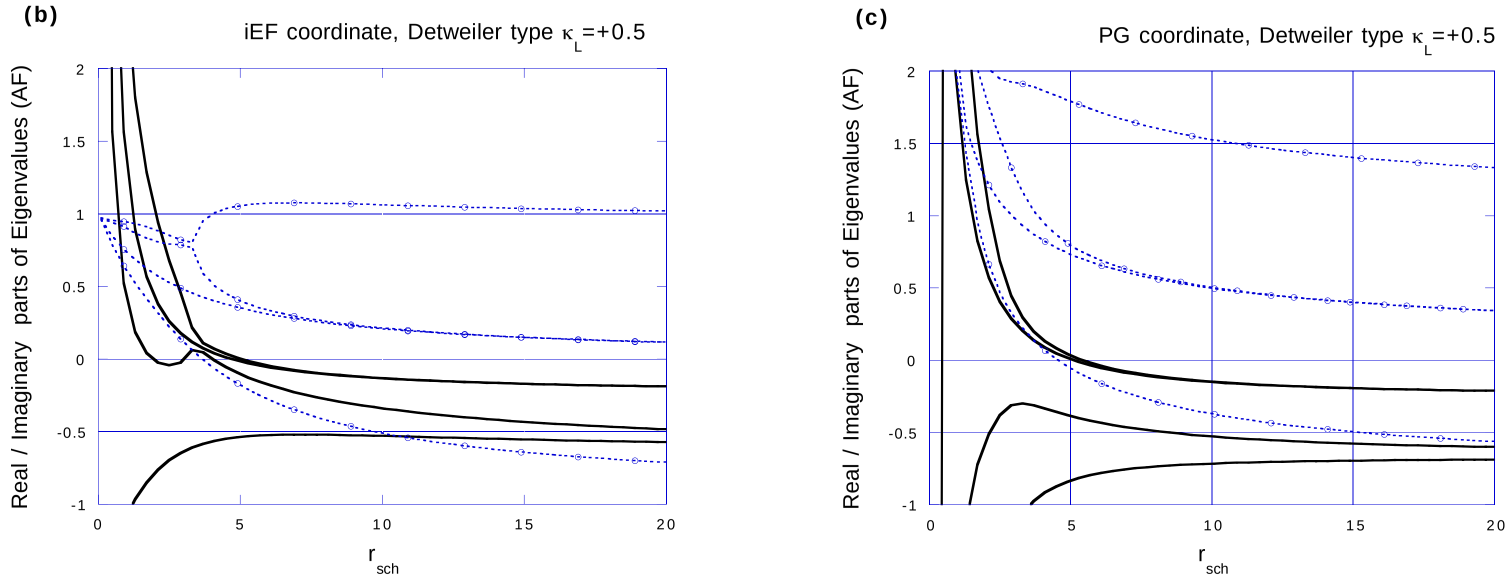


Figure 4: Similar comparison for Detweiler adjustments. $\kappa_L = +1/2$ for all plots.

No.	No. in Table.??	adjustment		1st?	Sch/iso coords.			iEF/PG coords.	
					TRS	real.	imag.	real.	imag.
0	0	–	no adjustments	yes	–	–	–	–	–
P-1	2-P	P_{ij}	$-\kappa_L\alpha^3\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-2	3	P_{ij}	$-\kappa_L\alpha\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-3	-	P_{ij}	$P_{rr} = -\kappa$ or $P_{rr} = -\kappa\alpha$	no	no	slightly enl.Neg.	not apparent	slightly enl.Neg.	not apparent
P-4	-	P_{ij}	$-\kappa\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-5	-	P_{ij}	$-\kappa\gamma_{rr}$	no	no	red. Pos./enl.Neg.	not apparent	red.Pos./enl.Neg.	not apparent
Q-1	-	$Q^k{}_{ij}$	$\kappa\alpha\beta^k\gamma_{ij}$	no	no	N/A	N/A	$\kappa \sim 1.35$ min. vals.	not apparent
Q-2	-	$Q^k{}_{ij}$	$Q^r{}_{rr} = \kappa$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
Q-3	-	$Q^k{}_{ij}$	$Q^r{}_{ij} = \kappa\gamma_{ij}$ or $Q^r{}_{ij} = \kappa\alpha\gamma_{ij}$	no	yes	red. abs vals.	not apparent	enl.Neg.	enl. vals.
Q-4	-	$Q^k{}_{ij}$	$Q^r{}_{rr} = \kappa\gamma_{rr}$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
R-1	1	R_{ij}	$\kappa_F\alpha\gamma_{ij}$	yes	yes	$\kappa_F = -1/4$ min. abs vals.		$\kappa_F = -1/4$ min. vals.	
R-2	4	R_{ij}	$R_{rr} = -\kappa_\mu\alpha$ or $R_{rr} = -\kappa_\mu$	yes	no	not apparent	not apparent	red.Pos./enl.Neg.	enl. vals.
R-3	-	R_{ij}	$R_{rr} = -\kappa\gamma_{rr}$	yes	no	enl. vals.	not apparent	red.Pos./enl.Neg.	enl. vals.
S-1	2-S	$S^k{}_{ij}$	$\kappa_L\alpha^2[3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l\alpha)\gamma_{ij}\gamma^{kl}]$	yes	no	not apparent	not apparent	not apparent	not apparent
S-2	-	$S^k{}_{ij}$	$\kappa\alpha\gamma^{lk}(\partial_l\gamma_{ij})$	yes	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
p-1	-	$p^k{}_{ij}$	$p^r{}_{ij} = -\kappa\alpha\gamma_{ij}$	no	no	red. Pos.	red. vals.	red. Pos.	enl. vals.
p-2	-	$p^k{}_{ij}$	$p^r{}_{rr} = \kappa\alpha$	no	no	red. Pos.	red. vals.	red.Pos/enl.Neg.	enl. vals.
p-3	-	$p^k{}_{ij}$	$p^r{}_{rr} = \kappa\alpha\gamma_{rr}$	no	no	makes 2 Neg.	enl. vals.	red. Pos. vals.	red. vals.
q-1	-	$q^{kl}{}_{ij}$	$q^{rr}{}_{ij} = \kappa\alpha\gamma_{ij}$	no	no	$\kappa = 1/2$ min. vals.	red. vals.	not apparent	enl. vals.
q-2	-	$q^{kl}{}_{ij}$	$q^{rr}{}_{rr} = -\kappa\alpha\gamma_{rr}$	no	yes	red. abs vals.	not apparent	not apparent	not apparent
r-1	-	$r^k{}_{ij}$	$r^r{}_{ij} = \kappa\alpha\gamma_{ij}$	no	yes	not apparent	not apparent	not apparent	enl. vals.
r-2	-	$r^k{}_{ij}$	$r^r{}_{rr} = -\kappa\alpha$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
r-3	-	$r^k{}_{ij}$	$r^r{}_{rr} = -\kappa\alpha\gamma_{rr}$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
s-1	2-s	$s^{kl}{}_{ij}$	$\kappa_L\alpha^3[\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}]$	no	no	makes 4 Neg.	not apparent	makes 4 Neg.	not apparent
s-2	-	$s^{kl}{}_{ij}$	$s^{rr}{}_{ij} = -\kappa\alpha\gamma_{ij}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.
s-3	-	$s^{kl}{}_{ij}$	$s^{rr}{}_{rr} = -\kappa\alpha\gamma_{rr}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.

Table 1: List of adjustments we tested in the Schwarzschild spacetime. The column of adjustments are nonzero multipliers. The effects to amplification factors (when $\kappa > 0$) are commented for each coordinate system and for real/imaginary parts of AFs, respectively. The 'N/A' means that there is no effect due to the coordinate properties; 'not apparent' means the adjustment does not change the AFs effectively according to our conjecture; 'enl./red./min.' means enlarge/reduce/minimize, and 'Pos./Neg.' means positive/negative, respectively. These judgements are made at the $r \sim O(10M)$ region on their $t = 0$ slice.

Example 5:
On Maximally-sliced hypersurfaces (standard ADM in Sch. coord.)

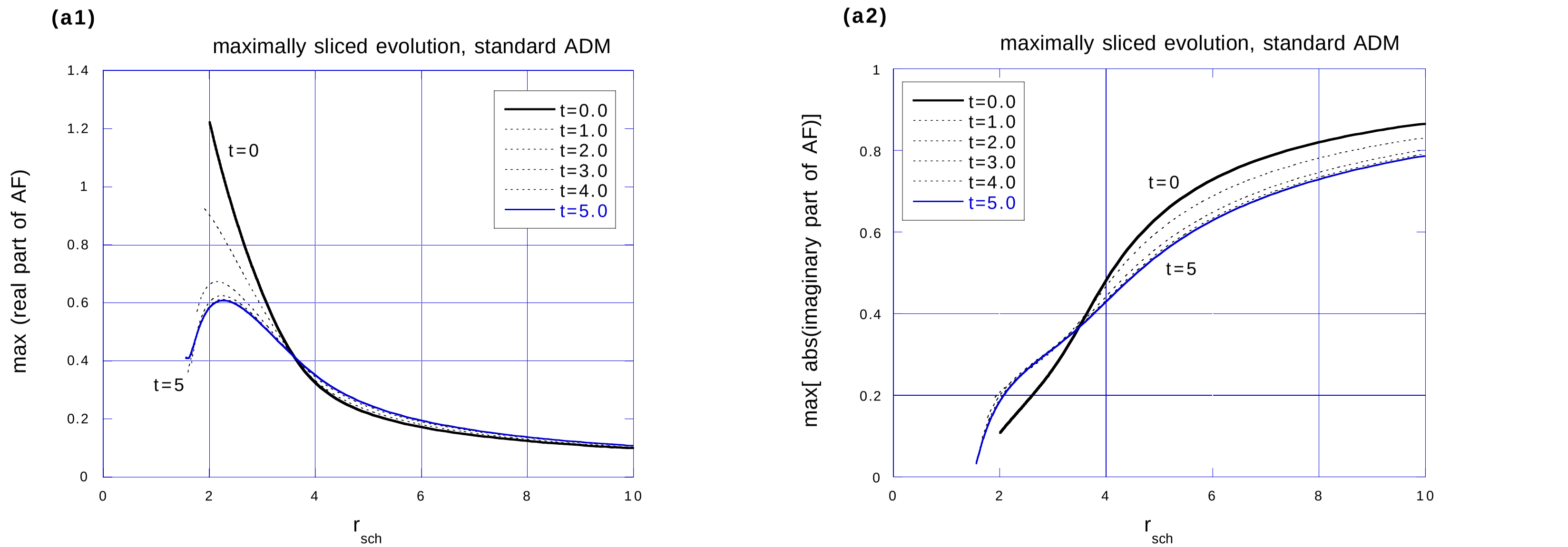


Figure 5: Amplification factors of snapshots of maximally-sliced evolving Schwarzschild spacetime. Fig (a1) and (a2) are of the standard ADM formulation (real and imaginary parts, respectively), Lines in (a1) are the largest (positive) AF on each time slice, while lines in (a2) are the maximum imaginary part of AF on each time slice. The lines start at $r_{min} = 2$ ($\bar{t} = 0$) and $r_{min} = 1.55$ ($\bar{t} = 5$).

"Einstein equations" are time-reversal invariant. So ...

Why all negative amplification factors (AFs) are available?

Explanation by the time-reversal invariance (TRI)

- the adjustment of the system I,

$$\text{adjust term to } \underbrace{\partial_t}_{(-)} \underbrace{K_{ij}}_{(-)} = \kappa_1 \underbrace{\alpha}_{(+)} \underbrace{\gamma_{ij}}_{(+)} \underbrace{\mathcal{H}}_{(+)}$$

preserves TRI. ... so the AFs remain zero (unchange).

- the adjustment by (a part of) Detweiler

$$\text{adjust term to } \underbrace{\partial_t}_{(-)} \underbrace{\gamma_{ij}}_{(+)} = -L \underbrace{\alpha}_{(+)} \underbrace{\gamma_{ij}}_{(+)} \underbrace{\mathcal{H}}_{(+)}$$

violates TRI. ... so the AFs can become negative.

Therefore

We can break the time-reversal invariant feature of the "ADM equations".

Advantages of modified ADM formulation

constraint propagation analysis of Baumgarte-Shapiro Shibata-Nakamura system

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OUTLINE

- Why Kyoto-type ADM works for long-term stable simulations?
- Much better formulations?

Refs:

G Yoneda and HS, Phys Rev D **63** (2001) 120419

HS and G Yoneda, Class. Quant. Grav. **19** (2002) 1027

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1 Introduction

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- M. Shibata and T. Nakamura, Phys. Rev. D **52**, 5428 (1995).
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2 BSSN equations and their constraint propagation equations

2.1 BSSN equations

The standard ADM formulation,

$$
\begin{aligned}
\partial_t^A \gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, \\
\partial_t^A K_{ij} &= \alpha R_{ij}^{ADM} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - D_iD_j\alpha + (D_i\beta^k)K_{kj} + (D_j\beta^k)K_{ki} + \beta^k D_k K_{ij}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}^{ADM} &= R^{ADM} + K^2 - K_{ij}K^{ij}, \\
\mathcal{M}_i^{ADM} &= D_j K^j{}_i - D_i K.
\end{aligned}
$$

The widely used BSSN notation is to introduce the variables $(\varphi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$

$$
\begin{aligned}
\varphi &= (1/12)\log(\det\gamma_{ij}), && (1) \\
\tilde{\gamma}_{ij} &= e^{-4\varphi}\gamma_{ij}, && (2) \\
K &= \gamma^{ij}K_{ij}, && (3) \\
\tilde{A}_{ij} &= e^{-4\varphi}(K_{ij} - (1/3)\gamma_{ij}K), && (4) \\
\tilde{\Gamma}^i &= \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk} && (5)
\end{aligned}
$$

- The new variable $\tilde{\Gamma}^i$ was introduced in order to calculate Ricci curvature more accurately.

$$\begin{aligned}
R^{ADM}_{ij} &= \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^l_{ij}\Gamma^k_{lk} - \Gamma^l_{kj}\Gamma^k_{li}, && (6)\\
R^{BSSN}_{ij} &= \tilde{R}_{ij} + R^{\varphi}_{ij}, && (7)\\
R^{\varphi}_{ij} &= -2\tilde{D}_i\tilde{D}_j\varphi - 2\tilde{\gamma}_{ij}\tilde{D}^k\tilde{D}_k\varphi + 4(\tilde{D}_i\varphi)(\tilde{D}_j\varphi) - 4\tilde{\gamma}_{ij}(\tilde{D}^k\varphi)(\tilde{D}_k\varphi),\\
\tilde{R}_{ij} &= -(1/2)\tilde{\gamma}^{lk}\partial_l\partial_k\tilde{\gamma}_{ij} + \tilde{\gamma}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + 2\tilde{\gamma}^{lm}\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{\gamma}^{lm}\tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj},
\end{aligned}$$

where $\tilde{D}_i$ is covariant derivative associated with $\tilde{\gamma}_{ij}$. These are weakly equivalent.

- the BSSN requires to impose the conformal factor as

$$\tilde{\gamma}(:= \det\tilde{\gamma}_{ij}) = 1, \qquad (8)$$

- Replacements of terms in the evolution equations using the constraints.

$$\begin{aligned}
\partial^B_t\varphi &= -(1/6)\alpha K + (1/6)\beta^i(\partial_i\varphi) + (\partial_i\beta^i), && (9)\\
\partial^B_t\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{ik}(\partial_j\beta^k) + \tilde{\gamma}_{jk}(\partial_i\beta^k) - (2/3)\tilde{\gamma}_{ij}(\partial_k\beta^k) + \beta^k(\partial_k\tilde{\gamma}_{ij}), && (10)\\
\partial^B_t K &= -D^iD_i\alpha + \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 + \beta^i(\partial_i K), && (11)\\
\partial^B_t\tilde{A}_{ij} &= -e^{-4\varphi}(D_iD_j\alpha)^{TF} + e^{-4\varphi}\alpha(R^{BSSN}_{ij})^{TF} + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{ik}\tilde{A}^k{}_j\\
&\quad +(\partial_i\beta^k)\tilde{A}_{kj} + (\partial_j\beta^k)\tilde{A}_{ki} - (2/3)(\partial_k\beta^k)\tilde{A}_{ij} + \beta^k(\partial_k\tilde{A}_{ij}), && (12)\\
\partial^B_t\tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} + 2\alpha(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - (2/3)\tilde{\gamma}^{ij}(\partial_j K) + 6\tilde{A}^{ij}(\partial_j\varphi)) - \partial_j(\beta^k(\partial_k\tilde{\gamma}^{ij})\\
&\quad -\tilde{\gamma}^{kj}(\partial_k\beta^i) - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)). && (13)
\end{aligned}$$

Constraints in BSSN system

The normal Hamiltonian and momentum constraints

$$\mathcal{H}^{BSSN} = R^{BSSN} + K^2 - K_{ij}K^{ij}, \tag{14}$$

$$\mathcal{M}_i^{BSSN} = \mathcal{M}_i^{ADM}, \tag{15}$$

Additionally, we regard the following three as the constraints:

$$\mathcal{G}^i = \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}, \tag{16}$$

$$\mathcal{A} = \tilde{A}_{ij}\tilde{\gamma}^{ij}, \tag{17}$$

$$\mathcal{S} = \tilde{\gamma} - 1, \tag{18}$$

Adjustments in evolution equations

$$\partial_t^B \varphi = \partial_t^A \varphi + (1/6)\alpha\mathcal{A} - (1/12)\tilde{\gamma}^{-1}(\partial_j\mathcal{S})\beta^j, \tag{19}$$

$$\partial_t^B \tilde{\gamma}_{ij} = \partial_t^A \tilde{\gamma}_{ij} - (2/3)\alpha\tilde{\gamma}_{ij}\mathcal{A} + (1/3)\tilde{\gamma}^{-1}(\partial_k\mathcal{S})\beta^k\tilde{\gamma}_{ij}, \tag{20}$$

$$\partial_t^B K = \partial_t^A K - (2/3)\alpha K\mathcal{A} - \alpha\mathcal{H}^{BSSN} + \alpha e^{-4\varphi}(\tilde{D}_j\mathcal{G}^j), \tag{21}$$

$$\partial_t^B \tilde{A}_{ij} = \partial_t^A \tilde{A}_{ij} + ((1/3)\alpha\tilde{\gamma}_{ij}K - (2/3)\alpha\tilde{A}_{ij})\mathcal{A} + ((1/2)\alpha e^{-4\varphi}(\partial_k\tilde{\gamma}_{ij}) - (1/6)\alpha e^{-4\varphi}\tilde{\gamma}_{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S}))\mathcal{G}^k + \alpha e^{-4\varphi}\tilde{\gamma}_{k(i}(\partial_{j)}\mathcal{G}^k) - (1/3)\alpha e^{-4\varphi}\tilde{\gamma}_{ij}(\partial_k\mathcal{G}^k) \tag{22}$$

$$\partial_t^B \tilde{\Gamma}^i = \partial_t^A \tilde{\Gamma}^i - ((2/3)(\partial_j\alpha)\tilde{\gamma}^{ji} + (2/3)\alpha(\partial_j\tilde{\gamma}^{ji}) + (1/3)\alpha\tilde{\gamma}^{ji}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) - 4\alpha\tilde{\gamma}^{ij}(\partial_j\varphi))\mathcal{A} - (2/3)\alpha\tilde{\gamma}^{ji}(\partial_j\mathcal{A}) + 2\alpha\tilde{\gamma}^{ij}\mathcal{M}_j - (1/2)(\partial_k\beta^i)\tilde{\gamma}^{kj}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) + (1/6)(\partial_j\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S}) + (1/3)(\partial_k\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) + (5/6)\beta^k\tilde{\gamma}^{-2}\tilde{\gamma}^{ij}(\partial_k\mathcal{S})(\partial_j\mathcal{S}) + (1/2)\beta^k\tilde{\gamma}^{-1}(\partial_k\tilde{\gamma}^{ij})(\partial_j\mathcal{S}) + (1/3)\beta^k\tilde{\gamma}^{-1}(\partial_j\tilde{\gamma}^{ji})(\partial_k\mathcal{S}). \tag{23}$$

A Full set of BSSN constraint propagation eqs.

$$\partial_t^{BS}\begin{pmatrix}\mathcal{H}^{BS}\\ \mathcal{M}_i\\ \mathcal{G}^i\\ \mathcal{S}\\ \mathcal{A}\end{pmatrix}=\begin{pmatrix}A_{11} & A_{12} & A_{13} & A_{14} & A_{15}\\ -(1/3)(\partial_i\alpha)+(1/6)\partial_i & \alpha K & A_{23} & 0 & A_{25}\\ 0 & \alpha\tilde\gamma^{ij} & 0 & A_{34} & A_{35}\\ 0 & 0 & 0 & \beta^k(\partial_k\mathcal{S}) & -2\alpha\tilde\gamma\\ 0 & 0 & 0 & 0 & \alpha K+\beta^k\partial_k\end{pmatrix}\begin{pmatrix}\mathcal{H}^{BS}\\ \mathcal{M}_j\\ \mathcal{G}^j\\ \mathcal{S}\\ \mathcal{A}\end{pmatrix}$$

$$\begin{aligned}
A_{11} &= +(2/3)\alpha K+(2/3)\alpha\mathcal{A}+\beta^k\partial_k\\
A_{12} &= -4e^{-4\varphi}\alpha(\partial_k\varphi)\tilde\gamma^{kj}-2e^{-4\varphi}(\partial_k\alpha)\tilde\gamma^{jk}\\
A_{13} &= -2\alpha e^{-4\varphi}\tilde A^k{}_j\partial_k-\alpha e^{-4\varphi}(\partial_j\tilde A_{kl})\tilde\gamma^{kl}-e^{-4\varphi}(\partial_j\alpha)\mathcal{A}-e^{-4\varphi}\beta^k\partial_k\partial_j-(1/2)e^{-4\varphi}\beta^k\tilde\gamma^{-1}(\partial_j\mathcal{S})\partial_k\\
&\quad +(1/6)e^{-4\varphi}\tilde\gamma^{-1}(\partial_j\beta^k)(\partial_k\mathcal{S})-(2/3)e^{-4\varphi}(\partial_k\beta^k)\partial_j\\
A_{14} &= 2\alpha e^{-4\varphi}\tilde\gamma^{-1}\tilde\gamma^{lk}(\partial_l\varphi)\mathcal{A}\partial_k+(1/2)\alpha e^{-4\varphi}\tilde\gamma^{-1}(\partial_l\mathcal{A})\tilde\gamma^{lk}\partial_k+(1/2)e^{-4\varphi}\tilde\gamma^{-1}(\partial_l\alpha)\tilde\gamma^{lk}\mathcal{A}\partial_k+(1/2)e^{-4\varphi}\tilde\gamma^{-1}\beta^m\tilde\gamma^{lk}\partial_m\partial_l\partial_k\\
&\quad -(5/4)e^{-4\varphi}\tilde\gamma^{-2}\beta^m\tilde\gamma^{lk}(\partial_m\mathcal{S})\partial_l\partial_k+e^{-4\varphi}\tilde\gamma^{-1}\beta^m(\partial_m\tilde\gamma^{lk})\partial_l\partial_k+(1/2)e^{-4\varphi}\tilde\gamma^{-1}\beta^i(\partial_j\partial_i\tilde\gamma^{jk})\partial_k\\
&\quad +(3/4)e^{-4\varphi}\tilde\gamma^{-3}\beta^i\tilde\gamma^{jk}(\partial_i\mathcal{S})(\partial_j\mathcal{S})\partial_k-(3/4)e^{-4\varphi}\tilde\gamma^{-2}\beta^i(\partial_i\tilde\gamma^{jk})(\partial_j\mathcal{S})\partial_k+(1/3)e^{-4\varphi}\tilde\gamma^{-1}\tilde\gamma^{pj}(\partial_j\beta^k)\partial_p\partial_k\\
&\quad -(5/12)e^{-4\varphi}\tilde\gamma^{-2}\tilde\gamma^{jk}(\partial_k\beta^i)(\partial_i\mathcal{S})\partial_j+(1/3)e^{-4\varphi}\tilde\gamma^{-1}(\partial_k\tilde\gamma^{ij})(\partial_j\beta^k)\partial_i-(1/6)e^{-4\varphi}\tilde\gamma^{-1}\tilde\gamma^{mk}(\partial_k\partial_l\beta^l)\partial_m\\
A_{15} &= (4/9)\alpha K\mathcal{A}-(8/9)\alpha K^2+(4/3)\alpha e^{-4\varphi}(\partial_i\partial_j\varphi)\tilde\gamma^{ij}+(8/3)\alpha e^{-4\varphi}(\partial_k\varphi)(\partial_l\tilde\gamma^{lk})+\alpha e^{-4\varphi}(\partial_j\tilde\gamma^{jk})\partial_k\\
&\quad +8\alpha e^{-4\varphi}\tilde\gamma^{jk}(\partial_j\varphi)\partial_k+\alpha e^{-4\varphi}\tilde\gamma^{jk}\partial_j\partial_k+8e^{-4\varphi}(\partial_l\alpha)(\partial_k\varphi)\tilde\gamma^{lk}+e^{-4\varphi}(\partial_l\alpha)(\partial_k\tilde\gamma^{lk})+2e^{-4\varphi}(\partial_l\alpha)\tilde\gamma^{lk}\partial_k\\
&\quad +e^{-4\varphi}\tilde\gamma^{lk}(\partial_l\partial_k\alpha)\\
A_{23} &= \alpha e^{-4\varphi}\tilde\gamma^{km}(\partial_k\varphi)(\partial_j\tilde\gamma_{mi})-(1/2)\alpha e^{-4\varphi}\tilde\Gamma^m_{kl}\tilde\gamma^{kl}(\partial_j\tilde\gamma_{mi})\\
&\quad +(1/2)\alpha e^{-4\varphi}\tilde\gamma^{mk}(\partial_k\partial_j\tilde\gamma_{mi})+(1/2)\alpha e^{-4\varphi}\tilde\gamma^{-2}(\partial_i\mathcal{S})(\partial_j\mathcal{S})-(1/4)\alpha e^{-4\varphi}(\partial_i\tilde\gamma_{kl})(\partial_j\tilde\gamma^{kl})+\alpha e^{-4\varphi}\tilde\gamma^{km}(\partial_k\varphi)\tilde\gamma_{ji}\partial_m\\
&\quad +\alpha e^{-4\varphi}(\partial_j\varphi)\partial_i-(1/2)\alpha e^{-4\varphi}\tilde\Gamma^m_{kl}\tilde\gamma^{kl}\tilde\gamma_{ji}\partial_m+\alpha e^{-4\varphi}\tilde\gamma^{mk}\tilde\Gamma_{ijk}\partial_m+(1/2)\alpha e^{-4\varphi}\tilde\gamma^{lk}\tilde\gamma_{ji}\partial_k\partial_l\\
&\quad +(1/2)e^{-4\varphi}\tilde\gamma^{mk}(\partial_j\tilde\gamma_{im})(\partial_k\alpha)+(1/2)e^{-4\varphi}(\partial_j\alpha)\partial_i+(1/2)e^{-4\varphi}\tilde\gamma^{mk}\tilde\gamma_{ji}(\partial_k\alpha)\partial_m\\
A_{25} &= -\tilde A^k{}_i(\partial_k\alpha)+(1/9)(\partial_i\alpha)K+(4/9)\alpha(\partial_iK)+(1/9)\alpha K\partial_i-\alpha\tilde A^k{}_i\partial_k\\
A_{34} &= -(1/2)\beta^k\tilde\gamma^{il}\tilde\gamma^{-2}(\partial_l\mathcal{S})\partial_k-(1/2)(\partial_l\beta^i)\tilde\gamma^{lk}\tilde\gamma^{-1}\partial_k+(1/3)(\partial_l\beta^l)\tilde\gamma^{ik}\tilde\gamma^{-1}\partial_k-(1/2)\beta^l\tilde\gamma^{in}(\partial_l\tilde\gamma_{mn})\tilde\gamma^{mk}\tilde\gamma^{-1}\partial_k\\
&\quad +(1/2)\beta^k\tilde\gamma^{il}\tilde\gamma^{-1}\partial_l\partial_k\\
A_{35} &= -(\partial_k\alpha)\tilde\gamma^{ik}+4\alpha\tilde\gamma^{ik}(\partial_k\varphi)-\alpha\tilde\gamma^{ik}\partial_k
\end{aligned}$$

3 Constraint propagation analysis in flat spacetime

3.1 BSSN constraint propagation equations

- The set of the constraint propagation equations, $\partial_t(\mathcal{H}^{BSSN}, \mathcal{M}_i, \mathcal{G}^i, \mathcal{A}, \mathcal{S})^T$?
- For the flat background metric $g_{\mu\nu} = \eta_{\mu\nu}$, the first order perturbation equations of (19)-(23):

$$\partial_t {}^{(1)}\varphi = -(1/6){}^{(1)}K + (1/6)\kappa_\varphi {}^{(1)}\mathcal{A} \tag{24}$$

$$\partial_t {}^{(1)}\tilde\gamma_{ij} = -2{}^{(1)}\tilde A_{ij} - (2/3)\kappa_{\tilde\gamma}\delta_{ij}{}^{(1)}\mathcal{A} \tag{25}$$

$$\partial_t {}^{(1)}K = -(\partial_j\partial_j {}^{(1)}\alpha) + \kappa_{K1}\partial_j {}^{(1)}\mathcal{G}^j - \kappa_{K2}{}^{(1)}\mathcal{H}^{BSSN} \tag{26}$$

$$\partial_t {}^{(1)}\tilde A_{ij} = {}^{(1)}(R_{ij}^{BSSN})^{TF} - {}^{(1)}(\tilde D_i \tilde D_j \alpha)^{TF} + \kappa_{A1}\delta_{k(i}(\partial_{j)}{}^{(1)}\mathcal{G}^k) - (1/3)\kappa_{A2}\delta_{ij}(\partial_k {}^{(1)}\mathcal{G}^k) \tag{27}$$

$$\partial_t {}^{(1)}\tilde\Gamma^i = -(4/3)(\partial_i {}^{(1)}K) - (2/3)\kappa_{\tilde\Gamma1}(\partial_i {}^{(1)}\mathcal{A}) + 2\kappa_{\tilde\Gamma2}{}^{(1)}\mathcal{M}_i \tag{28}$$

We express the adjustements as

$$\kappa_{adj} := (\kappa_\varphi, \kappa_{\tilde\gamma}, \kappa_{K1}, \kappa_{K2}, \kappa_{A1}, \kappa_{A2}, \kappa_{\tilde\Gamma1}, \kappa_{\tilde\Gamma2}). \tag{29}$$

- Constraint propagation equations at the first order in the flat spacetime:

$$\partial_t {}^{(1)}\mathcal{H}^{BSSN} = (\kappa_{\tilde\gamma} - (2/3)\kappa_{\tilde\Gamma1} - (4/3)\kappa_\varphi + 2)\,\partial_j\partial_j {}^{(1)}\mathcal{A} + 2(\kappa_{\tilde\Gamma2} - 1)(\partial_j {}^{(1)}\mathcal{M}_j), \tag{30}$$

$$\begin{aligned}\partial_t {}^{(1)}\mathcal{M}_i &= (-(2/3)\kappa_{K1} + (1/2)\kappa_{A1} - (1/3)\kappa_{A2} + (1/2))\,\partial_i\partial_j {}^{(1)}\mathcal{G}^j \\ &\quad + (1/2)\kappa_{A1}\partial_j\partial_j {}^{(1)}\mathcal{G}^i + ((2/3)\kappa_{K2} - (1/2))\,\partial_i {}^{(1)}\mathcal{H}^{BSSN},\end{aligned} \tag{31}$$

$$\partial_t {}^{(1)}\mathcal{G}^i = 2\kappa_{\tilde\Gamma2}{}^{(1)}\mathcal{M}_i + (-(2/3)\kappa_{\tilde\Gamma1} - (1/3)\kappa_{\tilde\gamma})(\partial_i {}^{(1)}\mathcal{A}), \tag{32}$$

$$\partial_t {}^{(1)}\mathcal{S} = -2\kappa_{\tilde\gamma}{}^{(1)}\mathcal{A}, \tag{33}$$

$$\partial_t {}^{(1)}\mathcal{A} = (\kappa_{A1} - \kappa_{A2})(\partial_j {}^{(1)}\mathcal{G}^j). \tag{34}$$

Effect of adjustments

No. in text.		Constraints (number of components) $\mathcal{H}$ (1)	$\mathcal{M}_i$ (3)	$\mathcal{G}^i$ (3)	$\mathcal{A}$ (1)	$\mathcal{S}$ (1)	Amplification Factors (AFs) in Minkowskii background
0.	standard ADM	use	use	-	-	-	$(0, 0, \Im, \Im)$
1.	BSSN no adjustment	use	use	use	use	use	$(0, 0, 0, 0, 0, 0, 0, \Im, \Im)$
2.	the BSSN	use+adj	use+adj	use+adj	use+adj	use+adj	$(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
3.	no $\mathcal{S}$ adjustment	use+adj	use+adj	use+adj	use+adj	use	no difference in flat background
4.	no $\mathcal{A}$ adjustment	use+adj	use+adj	use+adj	use	use+adj	$(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
5.	no $\mathcal{G}^i$ adjustment	use+adj	use+adj	use	use+adj	use+adj	$(0, 0, 0, 0, 0, 0, 0, \Im, \Im)$
6.	no $\mathcal{M}_i$ adjustment	use+adj	use	use+adj	use+adj	use+adj	$(0, 0, 0, 0, 0, 0, 0, \Re, \Re)$
7.	no $\mathcal{H}$ adjustment	use	use+adj	use+adj	use+adj	use+adj	$(0, 0, 0, , \Im, \Im, \Im, \Im, \Im, \Im)$
8.	ignore $\mathcal{G}^i$, $\mathcal{A}$, $\mathcal{S}$	use+adj	use+adj	-	-	-	$(0, 0, 0, 0)$
9.	ignore $\mathcal{G}^i$, $\mathcal{A}$	use+adj	use+adj	use+adj	-	-	$(0, \Im, \Im, \Im, \Im, \Im, \Im)$
10.	ignore $\mathcal{G}^i$	use+adj	use+adj	-	use+adj	use+adj	$(0, 0, 0, 0, 0, 0)$
11.	ignore $\mathcal{A}$	use+adj	use+adj	use+adj	-	use+adj	$(0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
12.	ignore $\mathcal{S}$	use+adj	use+adj	use+adj	use+adj	-	$(0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$

4 Proposals of Improved BSSN systems

(A) A system which has 8 pure imaginary AFs:

– One direction is to seek a possibility to reduce zero AFs than the standard BSSN case (No.2 in the previous section). Using the same set of adjustments in (24)-(28), AFs are written in general

$$AFs \;=\; (0, \pm\sqrt{-k^2\kappa_{A1}\kappa_{\tilde\Gamma 2}}\,\text{(2 pairs)}, \pm\text{complicated expression}, \pm\text{complicated expression}).$$

The terms in the first line certainly give four pure imaginary AFs (two positive and negative real pairs) if $\kappa_{A1}\kappa_{\tilde\Gamma 2} > 0\,(< 0)$.

– Keeping this in our mind, by choosing $\kappa_{adj} = (1,1,1,1,1,\kappa,1,1)$, we find

$$AFs \;=\; (0, \pm\sqrt{-k^2}\,\text{(2 pairs)}, \pm\sqrt{-k^2(2+\kappa+|\kappa-4|)/6}, \pm\sqrt{-k^2(2+\kappa-|\kappa-4|)/6},).$$

– Therefore the adjustment $\kappa_{adj} = (1,1,1,1,1,4,1,1)$ gives

$$AFs = (0, \pm\sqrt{-k^2}\,\text{(4 pairs)}),$$

which is one step advanced to the standard ADM according our guidelines.

– We note that such a system can be obtained in many ways, e.g. $\kappa_{adj} = (0,0,1,0,2,1,0,1/2)$ also gives four pairs of pure imaginary AFs.

(B) A system which has negative real AF:

– One criterion to obtain a decaying constraint mode (i.e. an asymptotically constrained system) is to adjust an evolution equation as it breaks time reversal symmetry.

– For example, we consider an additional adjustment to the BSSN equation as

$$\partial_t \tilde{\gamma}_{ij} = \partial_t^B \tilde{\gamma}_{ij} + \kappa_{SD} \alpha \tilde{\gamma}_{ij} \mathcal{H}^{BSSN},$$

which is a similar adjustment of simplified Detweiler-type.

– The first order constraint propagation equations on the flat background spacetime become

$$\begin{aligned}
\partial_t {}^{(1)}\mathcal{H}^{BSSN} &= \partial_j \partial_j {}^{(1)}\mathcal{A} - (3/2)\kappa_{SD} \partial_j \partial_j {}^{(1)}\mathcal{H}^{BSSN}, \\
\partial_t {}^{(1)}\mathcal{M}_i &= (1/6)\partial_i {}^{(1)}\mathcal{H}^{BSSN} + (1/2)\partial_j \partial_j {}^{(1)}\mathcal{G}^i, \\
\partial_t {}^{(1)}\mathcal{G}^i &= -\partial_i {}^{(1)}\mathcal{A} + (1/2)\kappa_{SD} \partial_i {}^{(1)}\mathcal{H}^{BSSN} + 2 {}^{(1)}\mathcal{M}_i, \\
\partial_t {}^{(1)}\mathcal{A} &= -(\partial_j \partial_j {}^{(1)}\alpha)^{TF} + ({}^{(1)}R_{jj}^{BSSN})^{TF}, \\
\partial_t {}^{(1)}\mathcal{S} &= -2 {}^{(1)}\mathcal{A} + 3\kappa_{SD} {}^{(1)}\mathcal{H}^{BSSN},
\end{aligned}$$

where we wrote only additional terms to (30)-(34).

– The amplification factors become

$$AF = (0\,(\times 2), \pm\sqrt{-k^2}\text{(3 pairs)}, (3/2)k^2\kappa_{SD}),$$

which the last one become negative real if $\kappa_{SD} < 0$.

(C) Combination of above (A) and (B)

– Naturally we next consider both two adjustments:

$$\begin{aligned} \partial_t \tilde{\gamma}_{ij} &= \partial_t^B \tilde{\gamma}_{ij} + \kappa_{SD} \alpha \tilde{\gamma}_{ij} \mathcal{H}^{BSSN} \\ \partial_t \tilde{A}_{ij} &= \partial_t^B \tilde{A}_{ij} - \kappa_8 \alpha e^{-4\varphi} \tilde{\gamma}_{ij} \partial_k \mathcal{G}^k \end{aligned}$$

where the second one produces the 8 pure imaginary AFs.

– The additional terms in the constraint propagation equations (30)-(34) are

$$\begin{aligned} \partial_t {}^{(1)}\mathcal{H}^{BSSN} &= \partial_j \partial_j {}^{(1)}\mathcal{A} - (3/2)\kappa_{SD} \partial_j \partial_j {}^{(1)}\mathcal{H}^{BSSN}, \\ \partial_t {}^{(1)}\mathcal{M}_i &= (1/6)\partial_i {}^{(1)}\mathcal{H}^{BSSN} + (1/2)\partial_j \partial_j {}^{(1)}\mathcal{G}^i - \kappa_8 \partial_i \partial_k {}^{(1)}\mathcal{G}^k, \\ \partial_t {}^{(1)}\mathcal{G}^i &= -\partial_i {}^{(1)}\mathcal{A} + (1/2)\kappa_{SD} \partial_i {}^{(1)}\mathcal{H}^{BSSN} + 2 {}^{(1)}\mathcal{M}_i, \\ \partial_t {}^{(1)}\mathcal{A} &= -3\kappa_8 \partial_k {}^{(1)}\mathcal{G}^k. \\ \partial_t {}^{(1)}\mathcal{S} &= -2 {}^{(1)}\mathcal{A} + 3\kappa_{SD} {}^{(1)}\mathcal{H}^{BSSN}, \end{aligned}$$

– We then obtain

$$AFs = (0, \pm\sqrt{-k^2} \text{ (3 pairs)}, (3/4)k^2\kappa_{SD} \pm \sqrt{k^2(-\kappa_8 + (9/16)k^2\kappa_{SD})})$$

which reproduce (A) if $\kappa_{SD} = 0, \kappa_8 = 1$, and (B) if $\kappa_8 = 0$. These AFs can become (0, pure imaginary (3 pairs), complex number with negative real part (1 pair)), with an appropriate combination of κ_8 and κ_{SD}.

5 Proposals of Improved BSSN systems (NEW!!)

5.1 TRS breaking adjustments

In order to break time reversal symmetry (TRS) of the evolution eqs, to adjust $\partial_t\phi$,$\partial_t\tilde{\gamma}_{ij}$,$\partial_t\tilde{\Gamma}^i$ using $\mathcal{S}, \mathcal{G}^i$, or to adjust $\partial_t K, \partial_t \tilde{A}_{ij}$ using $\tilde{\mathcal{A}}$.

$$\begin{aligned}
\partial_t\phi &= \partial_t^{BS}\phi + \kappa_{\phi\mathcal{H}}\alpha\mathcal{H}^{BS} + \kappa_{\phi\mathcal{G}}\alpha\tilde{D}_k\mathcal{G}^k + \kappa_{\phi\mathcal{S}1}\alpha\mathcal{S} + \kappa_{\phi\mathcal{S}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{S} \\
\partial_t\tilde{\gamma}_{ij} &= \partial_t^{BS}\tilde{\gamma}_{ij} + \kappa_{\tilde{\gamma}\mathcal{H}}\alpha\tilde{\gamma}_{ij}\mathcal{H}^{BS} + \kappa_{\tilde{\gamma}\mathcal{G}1}\alpha\tilde{\gamma}_{ij}\tilde{D}_k\mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{G}2}\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{S}1}\alpha\tilde{\gamma}_{ij}\mathcal{S} + \kappa_{\tilde{\gamma}\mathcal{S}2}\alpha\tilde{D}_i\tilde{D}_j\mathcal{S} \\
\partial_t K &= \partial_t^{BS}K + \kappa_{K\mathcal{M}}\alpha\tilde{\gamma}^{jk}(\tilde{D}_j\mathcal{M}_k) + \kappa_{K\tilde{\mathcal{A}}1}\alpha\tilde{\mathcal{A}} + \kappa_{K\tilde{\mathcal{A}}2}\alpha\tilde{D}^j\tilde{D}_j\tilde{\mathcal{A}} \\
\partial_t\tilde{A}_{ij} &= \partial_t^{BS}\tilde{A}_{ij} + \kappa_{A\mathcal{M}1}\alpha\tilde{\gamma}_{ij}(\tilde{D}^k\mathcal{M}_k) + \kappa_{A\mathcal{M}2}\alpha(\tilde{D}_{(i}\mathcal{M}_{j)}) + \kappa_{A\tilde{\mathcal{A}}1}\alpha\tilde{\gamma}_{ij}\tilde{\mathcal{A}} + \kappa_{A\tilde{\mathcal{A}}2}\alpha\tilde{D}_i\tilde{D}_j\tilde{\mathcal{A}} \\
\partial_t\tilde{\Gamma}^i &= \partial_t^{BS}\tilde{\Gamma}^i + \kappa_{\tilde{\Gamma}\mathcal{H}}\alpha\tilde{D}^i\mathcal{H}^{BS} + \kappa_{\tilde{\Gamma}\mathcal{G}1}\alpha\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}3}\alpha\tilde{D}^i\tilde{D}_j\mathcal{G}^j + \kappa_{\tilde{\Gamma}\mathcal{S}}\alpha\tilde{D}^i\mathcal{H}^{BS}
\end{aligned}$$

or in the flat background

$$\begin{aligned}
\partial_t^{ADJ(1)}\phi &= +\kappa_{\phi\mathcal{H}}{}^{(1)}\mathcal{H}^{BS} + \kappa_{\phi\mathcal{G}}\partial_k{}^{(1)}\mathcal{G}^k + \kappa_{\phi\mathcal{S}1}{}^{(1)}\mathcal{S} + \kappa_{\phi\mathcal{S}2}\partial_j\partial_j{}^{(1)}\mathcal{S} \\
\partial_t^{ADJ(1)}\tilde{\gamma}_{ij} &= +\kappa_{\tilde{\gamma}\mathcal{H}}\delta_{ij}{}^{(1)}\mathcal{H}^{BS} + \kappa_{\tilde{\gamma}\mathcal{G}1}\delta_{ij}\partial_k{}^{(1)}\mathcal{G}^k + (1/2)\kappa_{\tilde{\gamma}\mathcal{G}2}(\partial_j{}^{(1)}\mathcal{G}^i + \partial_i{}^{(1)}\mathcal{G}^j) + \kappa_{\tilde{\gamma}\mathcal{S}1}\delta_{ij}{}^{(1)}\mathcal{S} + \kappa_{\tilde{\gamma}\mathcal{S}2}\partial_i\partial_j{}^{(1)}\mathcal{S} \\
\partial_t^{ADJ(1)}K &= +\kappa_{K\mathcal{M}}\partial_j{}^{(1)}\mathcal{M}_j + \kappa_{K\tilde{\mathcal{A}}1}{}^{(1)}\tilde{\mathcal{A}} + \kappa_{K\tilde{\mathcal{A}}2}\partial_j\partial_j{}^{(1)}\tilde{\mathcal{A}} \\
\partial_t^{ADJ(1)}\tilde{A}_{ij} &= +\kappa_{A\mathcal{M}1}\delta_{ij}\partial_k{}^{(1)}\mathcal{M}_k + (1/2)\kappa_{A\mathcal{M}2}(\partial_i\mathcal{M}_j + \partial_j\mathcal{M}_i) + \kappa_{A\tilde{\mathcal{A}}1}\delta_{ij}\tilde{\mathcal{A}} + \kappa_{A\tilde{\mathcal{A}}2}\partial_i\partial_j\tilde{\mathcal{A}} \\
\partial_t^{ADJ(1)}\tilde{\Gamma}^i &= +\kappa_{\tilde{\Gamma}\mathcal{H}}\partial_i{}^{(1)}\mathcal{H}^{BS} + \kappa_{\tilde{\Gamma}\mathcal{G}1}{}^{(1)}\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}2}\partial_j\partial_j{}^{(1)}\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}3}\partial_i\partial_j{}^{(1)}\mathcal{G}^j + \kappa_{\tilde{\Gamma}\mathcal{S}}\partial_i{}^{(1)}\mathcal{S}
\end{aligned}$$

5.2 AFs with each adjustment

adjustment	AFs	expectations
$\kappa_{\phi\mathcal{H}}$	$(0, 0, \pm\sqrt{-k^2}(*3), 8\kappa_{\phi\mathcal{H}}k^2)$	$\kappa < 0$ makes 1 Neg.
$\kappa_{\phi\mathcal{G}}$	(long expressions)	$\kappa < 0$ makes 2 Neg. 1 Pos.
$\kappa_{\phi\mathcal{S}1}$	$(0, 0, 0, \pm\sqrt{-k^2}(*3))$	no effects?
$\kappa_{\phi\mathcal{S}2}$	$(0, 0, 0, \pm\sqrt{-k^2}(*3))$	no effects?
$\kappa_{\tilde{\gamma}\mathcal{H}}$	$(0, 0, \pm\sqrt{-k^2}(*2), (3/2)\kappa_{\tilde{\gamma}\mathcal{H}}k^2)$	$\kappa < 0$ makes 1 Neg.
$\kappa_{\tilde{\gamma}\mathcal{G}1}$	(long expressions)	$\kappa > 0$ makes 1 Neg.
$\kappa_{\tilde{\gamma}\mathcal{G}2}$	(long expressions)	$\kappa < 0$ makes 6 Neg. 1 Pos. GOOD!
$\kappa_{\tilde{\gamma}\mathcal{S}1}$	(long expressions)	$\kappa < 0$ makes 2 Neg. 1 Pos.
$\kappa_{\tilde{\gamma}\mathcal{S}2}$	(long expressions)	$\kappa \gg 0$ makes 2 Neg. 1 Pos.
$\kappa_{K\mathcal{M}}$	$(0, 0, 0, \pm\sqrt{-k^2}(*2), (1/3)\kappa_{K\mathcal{M}}k^2 \pm (1/3)\sqrt{k^2(-9 + k^2\kappa_{K\mathcal{M}}^2)})$	$\kappa < 0$ makes 2 Neg.
$\kappa_{K\tilde{\mathcal{A}}1}$	$(0, 0, 0, \pm\sqrt{-k^2}(*3))$	no effects?
$\kappa_{K\tilde{\mathcal{A}}2}$	$(0, 0, 0, \pm\sqrt{-k^2}(*3))$	no effects?
$\kappa_{A\mathcal{M}1}$	$(0, 0, \pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{M}1}k^2)$	$\kappa > 0$ makes 1 Neg.
$\kappa_{A\mathcal{M}2}$	(long expressions)	$\kappa > 0$ makes 7 Neg Excellent!!
$\kappa_{A\tilde{\mathcal{A}}1}$	$(0, 0, \pm\sqrt{-k^2}(*2), 3\kappa_{A\tilde{\mathcal{A}}1})$	$\kappa < 0$ makes 1 Neg.
$\kappa_{A\tilde{\mathcal{A}}2}$	$(0, 0, \pm\sqrt{-k^2}(*2), -\kappa_{A\tilde{\mathcal{A}}2}k^2)$	$\kappa > 0$ makes 1 Neg.
$\kappa_{\tilde{\Gamma}\mathcal{H}}$	$(0, 0, \pm\sqrt{-k^2}(*2), -\kappa_{A\tilde{\mathcal{A}}2}k^2)$	$\kappa > 0$ makes 1 Neg.
$\kappa_{\tilde{\Gamma}\mathcal{G}1}$	(long expressions)	$\kappa < 0$ makes 6 Neg. 1 Pos. GOOD!
$\kappa_{\tilde{\Gamma}\mathcal{G}2}$	(long expressions)	$\kappa > 0$ makes 6 Neg. 1 Pos. GOOD!
$\kappa_{\tilde{\Gamma}\mathcal{G}3}$	(long expressions)	$\kappa > 0$ makes 2 Neg. 1 Pos.
$\kappa_{\tilde{\Gamma}\mathcal{S}}$	(long expressions)	no effects? (all zeros?)

(D) A system which has 7 negative AFs: (NEW!)

$$\partial_t \tilde{A}_{ij} = \partial_t^{BSSN} \tilde{A}_{ij} + \kappa_{A\mathcal{M}2} \alpha (\tilde{D}_{(i} \mathcal{M}_{j)}) \tag{35}$$

$\kappa_{A\mathcal{M}2} > 0$, asymptotically constrained system.

Summary

Towards a stable and accurate formulation for numerical relativity

- Several reports say numerical stabilities depend on the formulations to apply, although they are mathematically equivalent.
- status = chaotic, many trials and errors
  We tried to understand the background systematically.
- Our proposal = "Evaluate eigenvalues of constraint propagation eqns"
  We give satisfactory conditions for stable evolutions.
  Fourier transformation allows to discuss lower-order terms.
- Our Observation = "Stability will change by adding constraints in RHS"
  We named "Adjusted System".
  Numerically confirmed in Maxwell system and Ashtekar system.
- Our Observation 2= The idea works even for the ADM formulation
  We explain the effective parameter range of Detweiler's system (1987).
  We proposed variety of adjustments.
- Our Observation 3= The idea works also for the BSSN formulation
  We explain why adjusting momentum constraints improves the stability.
  We proposed variety of adjustments.

Plans during this workshop

- Numerical tests of Adjusted ADM systems (myself)
  - models
    - linear wave, Brill wave evolution
    - Schwarzschild BH (if possible)
  - code `thorn ADMadjusted` which is a subset of `StandardEinstein/ADM`
  - status
    still coding 1D-code, not yet completed
    began coding thorn, not yet completed
- Numerical tests of Adjusted BSSN systems (volunteers?)
  - models
    - linear wave, Brill wave evolution
  - status: announcing
    Masaru Shibata tried system (B) for linear wave, and confirmed our prediction. But he seems not to try further