Cauchy boundaries in linearized gravitational theory

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ADM

Boundary conditions in linearized harmonic gravity

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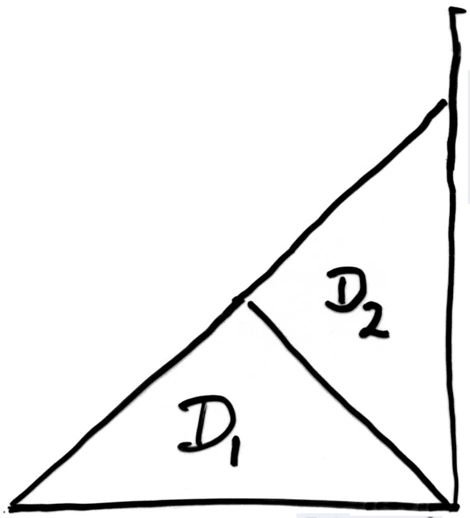
SYMMETRIC HYPERBOLIC



BOUNDARY DATA

BOUNDARY CONSTRAINTS

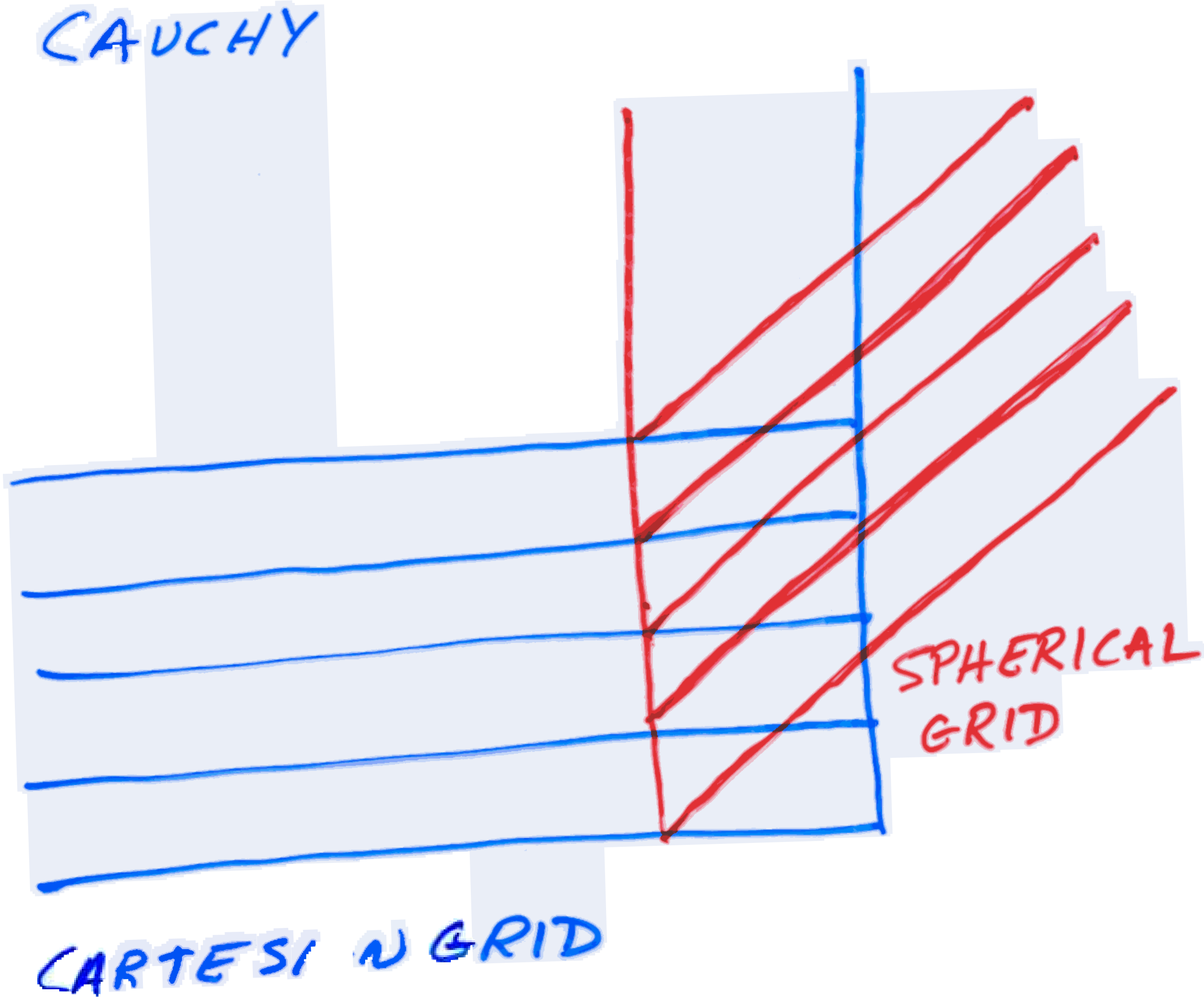
INITIAL DATA

CONSTRAINTS

Well Posed Initial-Boundary Evolution in General Relativity
B. Szylágyi & J. Winicour gr-qc 0205044

NONLINEAR EVOLUTION

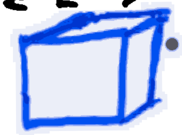
CAUCHY-CHARACTERISTIC MATCHING



LINEARIZED CCM WORKS!

- *Stage I:* Evolution on a 3-torus with random initial Cauchy data.

- *Stage II:* Evolution on a 2-torus with plane boundaries, i.e. $T^2 \times [-L, L]$, with random initial Cauchy data and random boundary data.

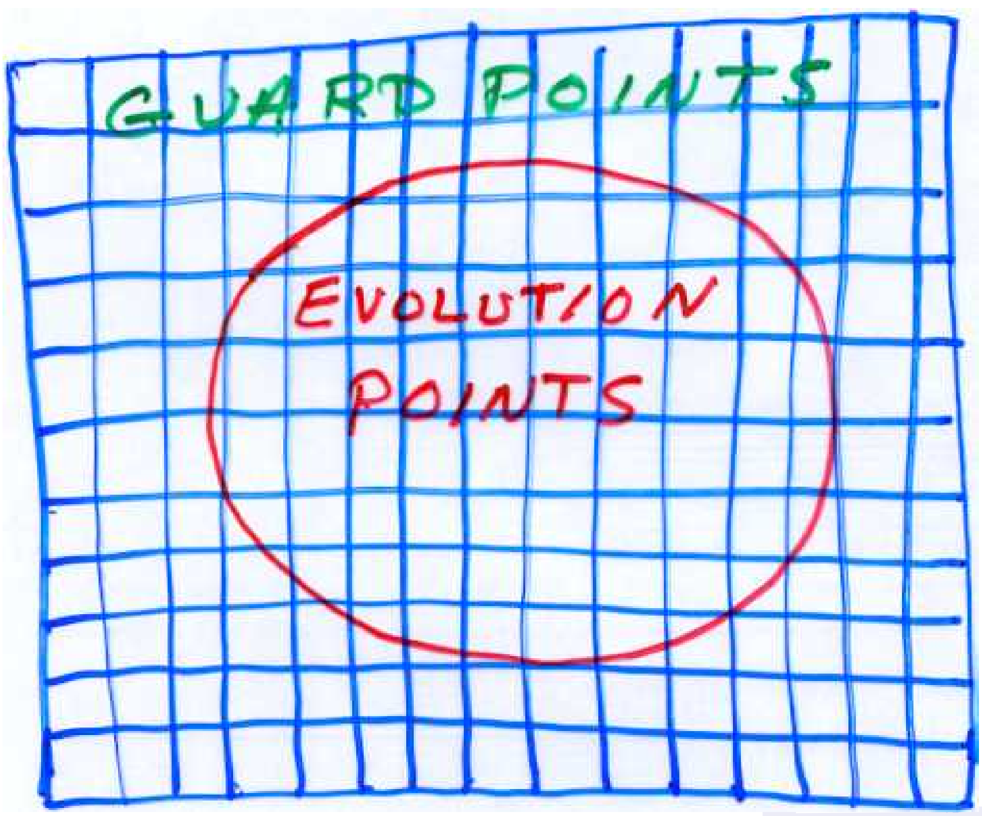


- *Stage III:* Evolution with a cubic boundary with random initial Cauchy data and random boundary data.

- *Stage IV:* Evolution with a spherical boundary with random initial Cauchy data and random free boundary data at all guard points.

Robustly stable if the ℓ_∞ norm of the Hamiltonian constraint shows no exponential growth in an evolution for 2000 crossing times on a uniform 48^3 spatial grid.

STAGE IV



CONVERGENCE IS ESSENTIAL.

BUT A CONVERGENCE TEST CONVERGES TO A ROBUST STABILITY TEST, UNLESS YOU HAVE AN IDEAL MACHINE.

Linearized ADM

Linearized metric

$$g_{\alpha\beta} \quad \eta_{\alpha\beta} + h_{\alpha\beta}$$

LAPSE =1 and SHIFT=0

$$h_{t\alpha} = 0$$

Evolution equations

$$\partial_t h_{ij} = -2K_{ij}$$
$$\partial_t K_{ij} = {}^{(3)}R_{ij},$$

Constraints

$$\mathcal{C} := G_{tt} = 0$$
$$\mathcal{C}_i := -G_{ti} = 0$$

Also considered 1-parameter family of evolution equations

$$\mathcal{E}_{ij} := R_{ij} + \frac{1}{2}\lambda\delta_{ij}\mathcal{C},$$

along with the constraints $\mathcal{C} = \mathcal{C}_i = 0$.

Stable constraint propagation requires $\lambda \geq -1$ (Fritelli). Bianchi identities imply

$$\ddot{\mathcal{C}} - (1+\lambda)\partial^k\partial_k\mathcal{C} = 0$$

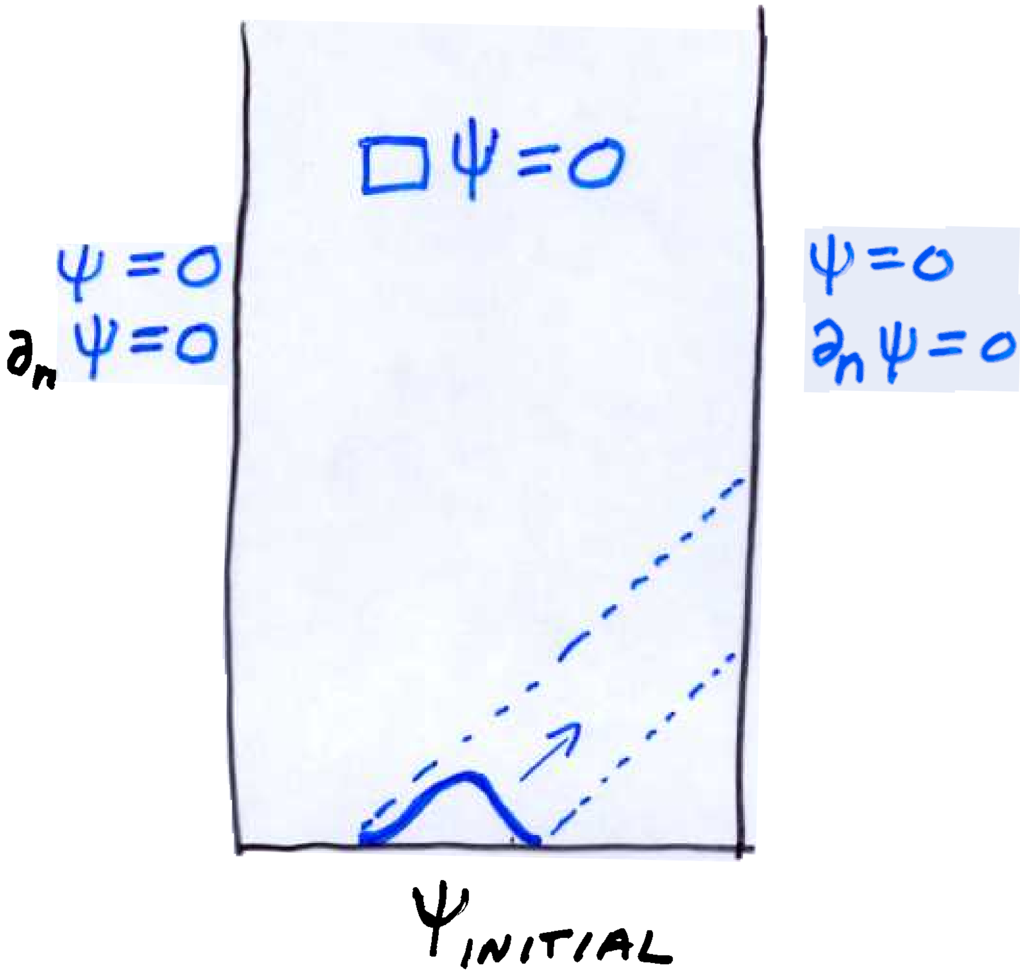
which is well-posed initial value problem for $\lambda \geq -$

ROBUSTLY STABLE Evolution Algorithms:
Standard leapfrog
Staggered leapfrog
Iterative Crank-Nicholson (2-iterations)

Boundary conditions: Illustrated by scalar wave data f on $z = const$ boundary.
Dirichlet data $\Phi(t,x,y,0) = f(t,x,y)$.
Neumann data $\partial_z\Phi(t,x,y,0) = f(t,x,y)$, implemented as a 3-point one-sided derivative.
Sommerfeld data $(\partial_t + \partial_z)\Phi(t,x,y,0) = f(t,x,y)$, implemented by 3-point spatial interpolation.

Prescription of Dirichlet boundary data for all spatial components of h_{ij} (or K_{ij}) of the ADM system ($\lambda = 0$) gives rise to an inconsistent evolution-boundary problem. The same is true for Neumann or Sommerfeld boundary data.

EQUIVALENT PROBLEM



NUMERICAL INSTABILITY ON SCALE OF ≈ 10 CROSSING TIMES

ROBUSTLY STABLE DIRICHLET EVOLUTION-BOUNDARY ALGORITHMS

Freely prescribe transverse-traceless data relative to boundary for K_{TT}

Example: $(K_{xx} - K_{yy})$ and K_{xy} for boundaries at $z = 0$ and $z = L$ We index the tangential (x, y) directions to boundary by x^A.

Given this TT boundary data, the boundary algorithm determines boundary values of the remaining components using the evolution equations and constraints.

We found five Dirichlet boundary algorithms that exhibit Stage II and III robust stability for the ICN evolution algorithm.
The algorithms update the boundary values of K_{ij}, with boundary values for h_{ij} updated by centered difference in time.

Given random initial data and random boundary data for the K_{TT}, all five boundary algorithms update the trace $K^x_x + K^y_y$ by

$$2G^z_z \quad 2\dot{K}^A_A + \partial^B \partial_B h^A_A \quad \partial_A \partial_B h^{AB} = 0.$$

In the best performing of the 5 cases, boundary values of K_{zz} are updated by

$$\dot{C}^z - \partial^z R^t_t \equiv \partial_z \dot{K}_{zz} + \partial_A \dot{K}^A_z \quad 0$$

and K^A_z by

$$2G^A_z \equiv -2\dot{K}^A_z - \partial_B(\partial^B h^A_z - \partial^A h^B_z).$$

In Stage III, the edges and corners of the cube are handled separately

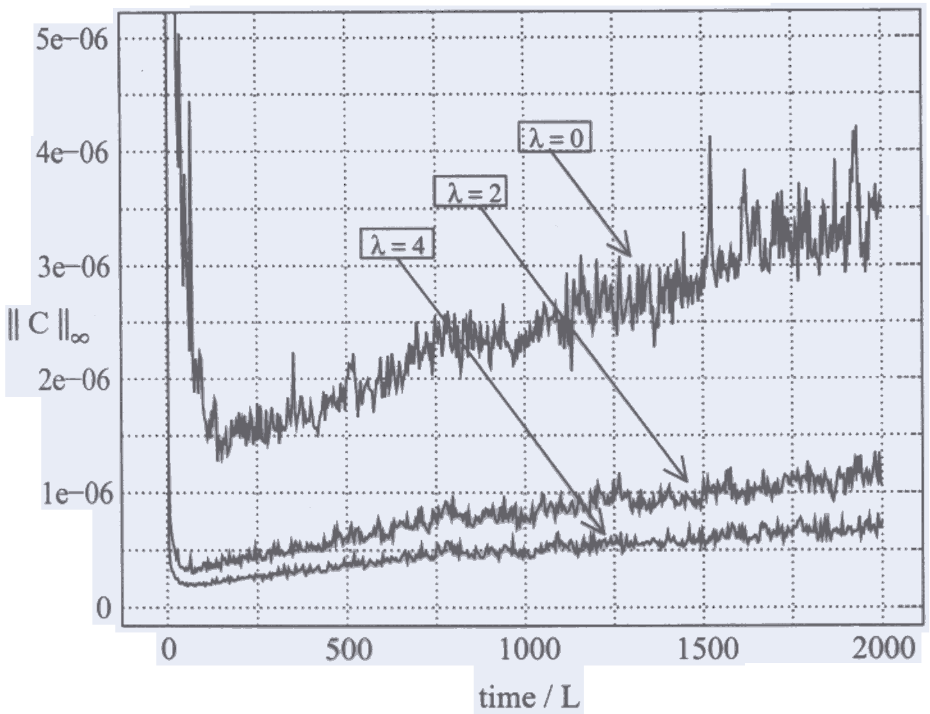


FIG. 3. Behavior of the Hamiltonian constraint for a Stage 3 test with cubic boundary.

IT'S POSSIBLE TO FLY BY THE SEAT OF YOUR PANTS, BUT . . .

LINEARIZED HARMONIC EVOLUTION

Linearized Einstein tensor in background Minkowskian coordinates:

$$^{\alpha\beta} \qquad h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} h$$

$$G^{\alpha\beta} = -\frac{1}{2}\Box\gamma^{\alpha\beta} - \partial^{(\alpha}\Gamma^{\beta)} + \frac{1}{2}\eta^{\alpha\beta}\partial_\mu\Gamma^\mu$$

Harmonic condition

$$\Gamma^\alpha \qquad -\partial_\beta\gamma^{\alpha\beta} = 0$$

Reduced harmonic evolution

Evolution variable

$$\kappa^{\alpha\beta} = \gamma^{\alpha\beta} - \frac{\lambda}{2}\eta^{\alpha\beta}\gamma$$

Unconstrained evolution equations

$$E^{ij} = G^{ij} - \frac{\lambda}{2}\eta^{ij}G = -\frac{1}{2}\Box\kappa^{ij} = 0$$

and harmonic condition $\Gamma^\alpha = 0$

Symmetric hyperbolic when $\Gamma^\alpha = 0$ is symmetric hyperbolic system for $\kappa^{t\alpha}$ Setting $\kappa^{ij} = 0$, $\Gamma^\alpha = 0$ implies

$$\frac{\lambda}{(3\lambda - 2)}\partial_i\partial^i)\kappa^{tt} = 0.$$

which is symmetric hyperbolic

$$\lambda \leq 0$$

or

$$\lambda > 2/3$$

Reduced Ricci evolution

Unconstrained evolution equations:

$$R^{ij} \qquad \frac{1}{2}\Box h^{ij} = 0$$

and harmonic conditions

$$\partial_t\phi + \partial_j h^{jt} + \frac{1}{2}\partial_t h^j_j = 0,$$
$$\partial^i\phi + \partial_t h^{it} - \frac{1}{2}\partial^i h^j_j + \partial_j h^{ij} = 0,$$

where $\phi = h^{tt}/2$.

Einstein's equations are satisfied only if $\mathcal{C} = \mathcal{C}^i \quad 0$