LINEARIZED HARMONIC EVOLUTION

Linearized Einstein tensor in background Minkowskian coordinates:

$$\gamma^{\alpha\beta} \quad h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} h$$

$$G^{\alpha\beta} \quad -\frac{1}{2}\Box\gamma^{\alpha\beta} - \partial^{(\alpha}\Gamma^{\beta)} + \frac{1}{2}\eta^{\alpha\beta}\partial_\mu\Gamma^\mu$$

Harmonic condition:

$$\Gamma^\alpha \quad -\partial_\beta\gamma^{\alpha\beta} \quad 0$$

Reduced harmonic evolution

Evolution variable:

$$\kappa^{\alpha\beta} \quad \gamma^{\alpha\beta} - \frac{\lambda}{2}\eta^{\alpha\beta}\gamma$$

Unconstrained evolution equations:

$$E^{ij} \quad G^{ij} - \frac{\lambda}{2}\eta^{ij}\delta G \quad -\frac{1}{2}\Box\kappa^{ij} = 0$$

and harmonic condition $\Gamma^\alpha = 0$

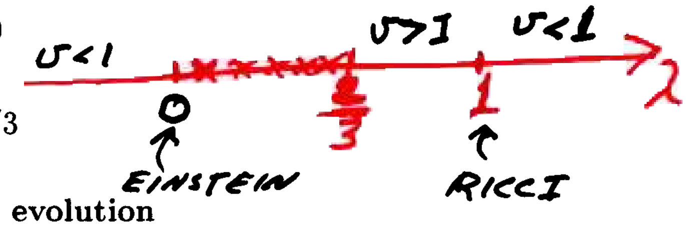
Symmetric hyperbolic when $\Gamma^\alpha = 0$ is symmetric hyperbolic system for $\kappa^{t\alpha}$ Setting $\kappa^{ij} = 0$, $\Gamma^\alpha = 0$ implies

$$(\partial_t^2 - \frac{\lambda}{(3\lambda - 2)}\partial_i\partial^i)\kappa^{tt} \quad 0.$$

which is symmetric hyperbolic when

$$\lambda \le 0$$

$$\lambda > 2/3$$



Reduced Ricci evolution

Unconstrained evolution equations:

$$R^{ij} = -\frac{1}{2}\Box h^{ij} = 0$$

and harmonic conditions

$$\partial_t\phi + \partial_j h^{jt} + \frac{1}{2}\partial_t h^j_j = 0,$$

$$\partial^i\phi + \partial_t h^{it} - \frac{1}{2}\partial^i h^j_j + \partial_j h^{ij} = 0,$$

where $\phi = h^{tt}/2$.

Einstein's equations are satisfied only if $C \quad C^i = 0$.

SYMMETRIC HYPERBOLIC FORM OF REDUCED RICCI SYSTEM

Auxiliary variables:

$$T^{ij} \quad \partial_t h^{ij}, X^{ij} \quad \partial_x h^{ij}, Y^{ij} = \partial_y h^{ij}, Z^{ij} = \partial_z h^{ij}$$

34-dimensional evolution vector:

$$u = \left(h^{ij}, T^{ij}, X^{ij}, Y^{ij}, Z^{ij}, h^{it}, \phi = \frac{1}{2} h^{tt} \right)$$

Symmetric hyperbolic unconstrained evolution system:

$$\partial_t u + A^i \, \partial_i u = B \, u$$

where

$$A^z \quad \begin{pmatrix}
0_{6\times6} & 0_{6\times6} & 0_{6\times6} & 0_{6\times6} & 0_{6\times6} & 0_{1\times6} & 0_{1\times6} & 0_{1\times6} & 0_{1\times6} \\
0_{6\times6} & 0_{6\times6} & 0_{6\times6} & 0_{6\times6} & -\mathbf{I}_{6\times6} & 0_{1\times6} & 0_{1\times6} & 0_{1\times6} & 0_{1\times6} \\
0_{6\times6} & 0_{6\times6} & 0_{6\times6} & 0_{6\times6} & 0_{6\times6} & 0_{1\times6} & 0_{1\times6} & 0_{1\times6} & 0_{1\times6} \\
0_{6\times6} & 0_{6\times6} & 0_{6\times6} & 0_{6\times6} & 0_{6\times6} & 0_{1\times6} & 0_{1\times6} & 0_{1\times6} & 0_{1\times6} \\
0_{6\times6} & -\mathbf{I}_{6\times6} & 0_{6\times6} & 0_{6\times6} & 0_{6\times6} & 0_{1\times6} & 0_{1\times6} & 0_{1\times6} & 0_{1\times6} \\
0_{6\times1} & 0_{6\times1} & 0_{6\times1} & 0_{6\times1} & 0_{6\times1} & 0_{1\times1} & 0_{1\times1} & 0_{1\times1} & 0_{1\times1} \\
0_{6\times1} & 0_{6\times1} & 0_{6\times1} & 0_{6\times1} & 0_{6\times1} & 0_{1\times1} & 0_{1\times1} & 0_{1\times1} & 0_{1\times1} \\
0_{6\times1} & 0_{6\times1} & 0_{6\times1} & 0_{6\times1} & 0_{6\times1} & 0_{1\times1} & 0_{1\times1} & 0_{1\times1} & +\mathbf{I}_{1\times1} \\
0_{6\times1} & 0_{6\times1} & 0_{6\times1} & 0_{6\times1} & 0_{6\times1} & 0_{1\times1} & 0_{1\times1} & +\mathbf{I}_{1\times1} & 0_{1\times1}
\end{pmatrix}$$

Maximally dissipative boundary:

$$^{T}u A^z u \leq 0.$$

This inequality ensures a well posed initial-boundary value problem

Eigenvalues and eigenvectors of A^z

with multiplicity 7 and eigenvectors

$$\left(\partial_t - \partial_z\right) h^{ij} \qquad \phi + h^{zt};$$

1, with multiplicity 7 and eigenvectors

$$\left(\partial_t + \partial_z\right) h^{ij} \qquad \phi - h$$

0, with multiplicity 20 and eigenvectors

$$h^{ij}, \quad \partial_x h^{ij}, \quad \partial_y h^{ij}, \quad h^{xt}, h^{yt}$$

MAXIMALLY DISSIPATIVE DATA

In the eigen-basis of A^z, u decomposes into

$$u = {}^T(u_-, u_0, u_+$$

with

$$\begin{aligned} u_{\cdot} &= {}^T(T^{xx} + Z^{xx}, \ldots, T^{zz} + Z^{zz}, \phi - h^{zt}) \\ &\quad {}^T(h^{xx}, \ldots, h^{zz}, X^{xx}, \ldots, X^{zz}, Y^{xx}, \ldots \; Y^{zz}, h^{xt}, h^{yt}) \\ &\quad {}^T(T^{xx} - Z^{xx}, \ldots, T^{zz} - Z^{zz}, \phi + h^{zt}) \end{aligned}$$

Free boundary data q can be prescribed in the form

$$u_+ - H u_- = q$$

where H is a 7×7 matrix and q is a column vector field.

Maximally dissipative condition:

$${}^T u \, {}^T H \, H \, u \leq {}^T u \, u,$$

Three simplest choices of H:

$-\mathbf{I}_{7\times 7}$ corresponding to Neumann data for h^{ij} and Dirichlet data for h^{zt}

$0_{7\times 7}$ corresponding to Sommerfeld data for h^{ij} and Dirichlet data for $\phi - h^{zt}$

$\mathbf{I}_{7\times 7}$ corresponding to Dirichlet data for $\partial_t h^{ij}$ and ϕ.

$\Box \Phi + f(\Phi) = 0$ $\qquad$ $u = (\Phi, \nabla_\alpha \phi)$

FLUX $\quad F^z = TZ \geq 0$

Dirichlet $\quad T = \partial_t \Phi = 0$

Neumann $\quad Z = \partial_z \phi = 0$

Sommerfeld $\quad T + Z = 0$

Use maximally dissipative boundary data for 2nd spatial derivative form of wave equation with a 2-iteration Crank-Nicholson evolution.

First you establish robust stability of the constraint-free reduced system.

SATISFYING EINSTEIN'S EQUATION

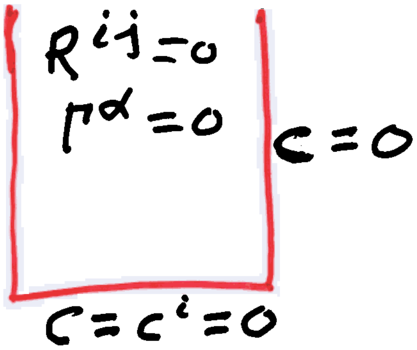
Reduced Ricci evolution

Unconstrained evolution equations:$R^{ij} = -\frac{1}{2}\Box h^{ij} = 0$ and $\Gamma^\alpha = 0$.

Einstein's equations are satisfied by solutions of the reduced system if and only if $\mathcal{C} = \mathcal{C}^i = 0$.

Bianchi identities:

$$\partial_t \mathcal{C}^i + \partial^i \mathcal{C} + \partial_j R^{ij} = 0$$
$$\partial_t \mathcal{C} + \partial_j \mathcal{C}^j = 0,$$



When $R^{ij} = 0$, they imply

$$\Box \mathcal{C} = 0$$

As a result:

Einstein's equations are satisfied if $R^{ij} = \Gamma^\alpha = 0$, $\mathcal{C} = \mathcal{C}^i = 0$ **initially and** $\mathcal{C} = 0$ **on the boundary).**

Modulo the evolution equations, $\mathcal{C} = 0$ **is equivalent to the boundary constraint**

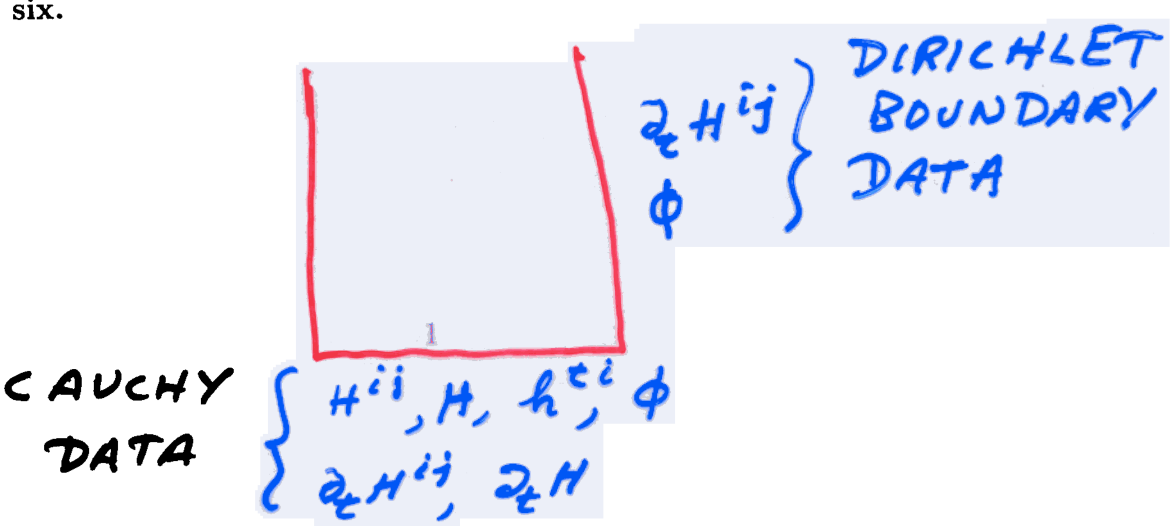
$$\frac{2}{3}\partial_t^2 H = \partial_i \partial_j H^{ij},$$

where $H = h_i^i$ **and** $H^{ij} = h^{ij} - \frac{1}{3}\delta^{ij} h_k^k$.

Well posed initial-boundary value problem for a solution satisfying the constraints:

Prescribe initial Cauchy data that satisfies the constraints for H^{ij}, H, ϕ **and** h^{it}, **and free boundary data for** H^{ij} **and** ϕ. **The system of wave equations then determines** H^{ij}, **which via the boundary constraint provides Dirichlet boundary data for** H **as a solution of the wave equation. The remaining fields** ϕ **and** h^{it} **are evolved as a symmetric hyperbolic subsystem.**

Note: the boundary constraint reduces the free boundary data from seven functions to six.



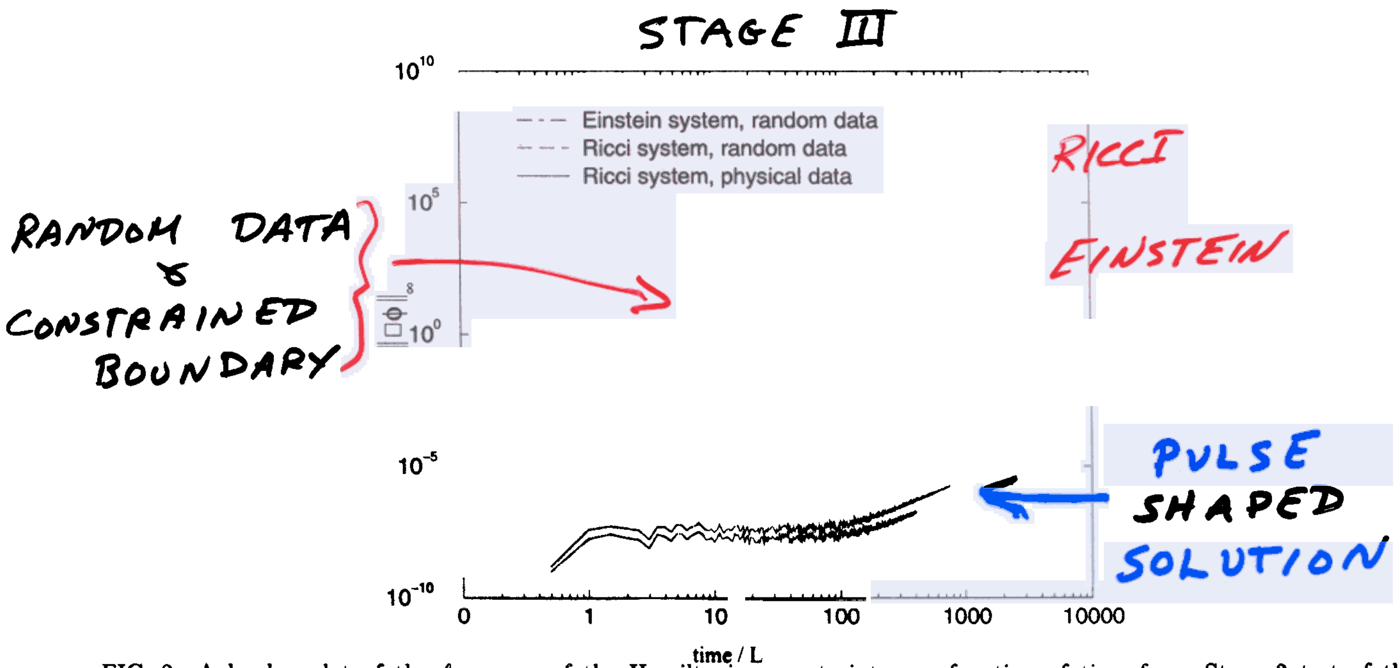


FIG. 2. A log-log plot of the ℓ_∞ norm of the Hamiltonian constraint as a function of time for a Stage 3 test of the evolution-boundary algorithm of the Einstein system and of the Ricci system with constrained boundary. The upper two curves correspond to stability tests (random data), while the lower two curves indicate performance tests (physical data).

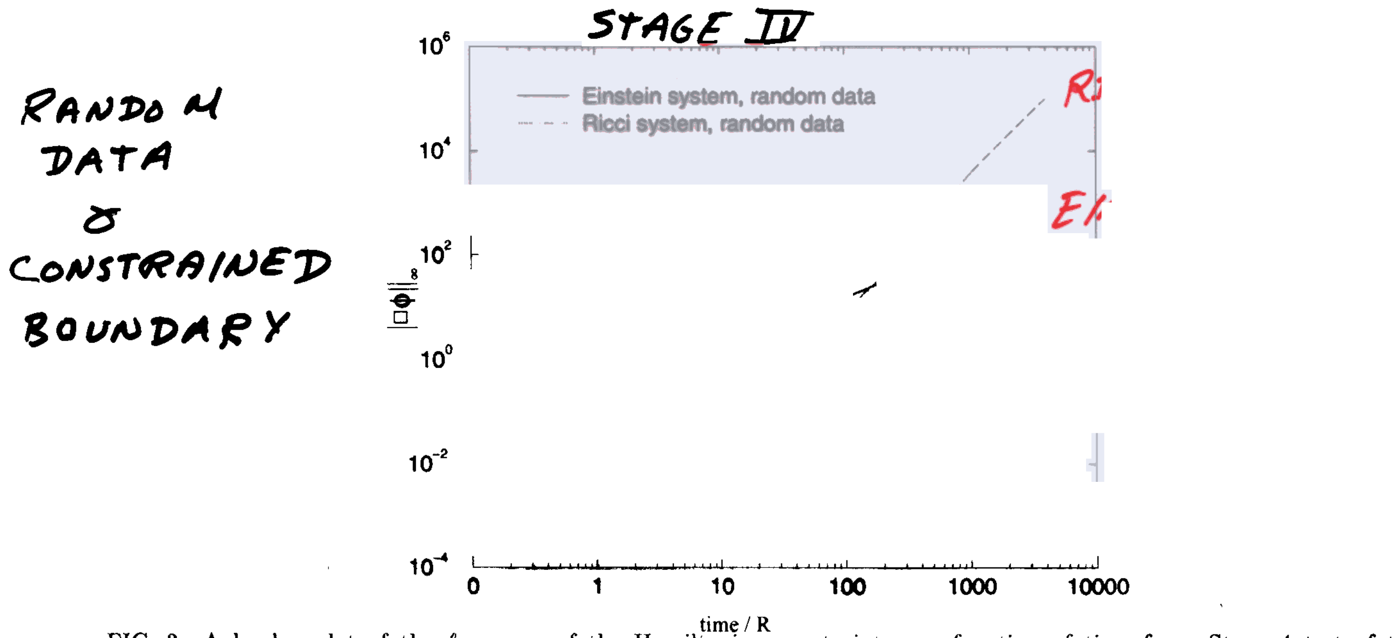


FIG. 3. A log-log plot of the ℓ_∞ norm of the Hamiltonian constraint as a function of time for a Stage 4 test of the evolution-boundary algorithm of the Einstein system and the Ricci system with constrained boundary.