Stability of the KST Equations

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- Kidder-Scheel-Teukolsky (KST) give a rather general 12-parameter family of first-order hyperbolic formulations of the Einstein evolution equations, which can be represented in the form,

$$\partial_t u^\alpha + A^{k\alpha}{}_\beta \partial_k u^\beta = F^\alpha,$$

  where, e.g. $u^\alpha = \{g_{ij}, p_{ij}, m_{kij}\}$, is the collection of 30 dynamical fields, and $A^{k\alpha}{}_\beta$ and F^α depend on u^α and on the 12 parameters.
- The stability of numerical evolutions of these equations has been shown to depend on the parameters that determine the specific representation.
- Can we develop tools that allow us to predict which representations are the most stable without having to solve the full non-linear equations?

- The California Citrus Growers Association asks this scholarly meeting to consider the historical and philosophical question:

"Is it possible that Einstein preferred Oranges to Apples?"

- KST introduce 4 parameters, $(\gamma, \zeta, \eta, \chi)$, by adding multiples of the constraints to the standard ADM 3+1 equations:

$$\begin{aligned} \partial_t g_{ij} &= \ldots, \\ \partial_t K_{ij} &= \ldots + \gamma N g_{ij}\,\mathcal{C} + \zeta N g^{ab}\mathcal{C}_{a(ij)b}, \\ \partial_t m^0_{kij} &= \ldots + \tfrac{1}{2}\eta N g_{k(i}\,\mathcal{C}_{j)} + \tfrac{1}{2}\chi N g_{ij}\,\mathcal{C}_k, \end{aligned}$$

where N is the lapse, $m^0_{kij} = \frac{1}{2}\,\partial_k g_{ij}$, and $\mathcal{C}_{\mathcal{A}}$, are the constraints:

$$\begin{aligned} \mathcal{C} &= \tfrac{1}{2}\,(R - K_{ab}K^{ab} + K^2), \\ \mathcal{C}_i &= \nabla_a K^a_i - \nabla_i K, \\ \mathcal{C}_{klij} &= \partial_k\, m^0_{lij} - \partial_l\, m^0_{kij}. \end{aligned}$$

- Introduce a fifth parameter σ by assuming that the lapse is determined from a function $Q(x^i, t)$ specified on spacetime a priori:

$$N = g^{\sigma} Q,$$

where g is the determinant of g_{ij}.

- Introduce 7 additional parameters, $(\hat{z}, \hat{k}, \hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e})$, by re-defining the dynamical fields:

$$\begin{aligned} p_{ij} &= K_{ij} + \hat{z}\, g_{ij} K, \\ m_{kij} &= \hat{k}\, m^0_{kij} + \hat{e}\, m^0_{(ij)k} + g_{ij}(\hat{a}\, m^0_{ka}{}^a + \hat{b}\, m^{0a}_a{}_k) + g_{k(i}(\hat{c}\, m^0_{j)a}{}^a + \hat{d}\, m^{0a}_a{}_{j)}). \end{aligned}$$

- This second set of parameters define a “linear” transformation on the fundamental dynamical fields, $u^\alpha_0 \equiv \{g_{ij}, K_{ij}, m^0_{kij}\}$:

$$u^\alpha \equiv \{g_{ij}, p_{ij}, m_{kij}\} = T^\alpha{}_\beta u^\beta_0,$$

- The evolution equations for the new fields are simple transformations of the fundamental equations:

$$\partial_t u^\alpha + A^{k\alpha}{}_\beta \partial_k u^\beta = \partial_t u^\alpha + T^\alpha{}_\mu A^{k\mu}_0{}_\nu T^{-1\,\nu}{}_\beta \partial_k u^\beta = F^\alpha.$$

- The characteristic speeds, v, of the transformed equations depend therefore only on the five *dynamical* parameters, $(\gamma, \zeta, \eta, \chi, \sigma)$:

$$0 = \det(v\, \delta^\alpha{}_\beta - n_k A^{k\,\alpha}{}_\beta) = \det(v\, \delta^\alpha{}_\beta - n_k A^{k\alpha}_0{}_\beta).$$

- Can we construct a “metric”, $S_{\alpha\beta}$, on the space of fields with the following properties?
  - $S_{\alpha\beta}u^\alpha u^\beta \geq 0 \quad \forall \quad u^\alpha,$
  - $S_{\alpha\mu}A^{k\mu}{}_\beta \equiv A^k_{\alpha\beta} = A^k_{\beta\alpha} \quad \forall \quad k.$

  That is, are the KST equations symmetric hyperbolic?
- YES! We have explicitly constructed such $S_{\alpha\beta}$ for the 9-parameter family of KST equations having only “physical” characteristic speeds, $v = \{0, \pm 1\}$. The resulting $S_{\alpha\beta}$ are positive definite iff $0 > \zeta > -\frac{5}{3}$.
- For these KST equations there exists a 4-parameter family of $S_{\alpha\beta}$ having the form:

$$\begin{aligned} dS^2 \equiv S_{\alpha\beta}du^\alpha du^\beta \quad = \quad & \left(a_1 - \tfrac{1}{3}\,a_2\right) g^{ij}g^{kl}dg_{ij}dg_{kl} + a_2 g^{ik}g^{jl}dg_{ij}dg_{kl} \\ & + \left(b_1 - \tfrac{1}{3}\,b_2\right) g^{ij}g^{kl}dp_{ij}dp_{kl} + b_2 g^{ik}g^{jl}dp_{ij}dp_{kl} \\ & + \ldots \end{aligned}$$

- These $S_{\alpha\beta}$ may be used to construct “energy norms” which in turn can be used to measure and even predict the stability of the equations.

- Consider the equations for the evolution of small perturbations δu^α about some fixed background which satisfies the evolution and constraint eqs:

$$\partial_t \delta u^\alpha + A^{k\alpha}{}_\beta \partial_k \delta u^\beta = F^\alpha{}_\beta \delta u^\beta.$$

- Next define an "energy" and "current" associated with the perturbations:

$$\begin{aligned} \delta E &= S_{\alpha\beta} \delta u^\alpha \delta u^\beta \\ \delta E^k &= A^k_{\alpha\beta} \delta u^\alpha \delta u^\beta \end{aligned}$$

- Compute the time evolution of δE:

$$\partial_t \delta E + \nabla_k \delta E^k = C_{\alpha\beta} \delta u^\alpha \delta u^\beta.$$

- Integrating over a spatial slice determines the growth rate of an exponentially growing solution:

$$\partial_t \int \delta E d^3x = \frac{2}{\tau} \int \delta E d^3x = \int C_{\alpha\beta} \delta u^\alpha \delta u^\beta d^3x - \oint n_k \delta E^k d^2x.$$

- We have verified that $1/\tau$ obtained from this energy expression is identical to the growth rate of the constraint violations found numerically by KST:

$$\frac{2}{\tau}\int \delta E d^3x = \int C_{\alpha\beta}\delta u^\alpha \delta u^\beta d^3x - \oint n_k \delta E^k d^2x.$$

- This energy expression is an identity for exact solutions to the perturbed equations δu^α. Can we obtain useful estimates of the growth rate of the most unstable mode by using approximate expressions for the eigenfunction δu^α?

- The simplest assumption is to take δu^α to be spatially constant: $\partial_k \delta u^\alpha = 0$.

- In this case the energy expression become:

$$\frac{2}{\tau}\int \delta E d^3x = \int \left[C_{\alpha\beta} - (\sqrt{g})^{-1}\partial_k\left(\sqrt{g}A^k_{\alpha\beta}\right)\right]\delta u^\alpha \delta u^\beta d^3x \equiv \int \bar{C}_{\alpha\beta}\delta u^\alpha \delta u^\beta d^3x.$$

- The maximum growth rate $1/\tau$ for such δu^α is just (half) the maximum eigenvalue of $\bar{C}_{\alpha\beta}$.

- How do the energy δE and related quantities transform under the kinematical parameters: $(\hat{z}, \hat{k}, \hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e})$?
- Recall that these parameters change the dynamical fields via a “linear” transformation: $u^\alpha = T^\alpha{}_\beta u_0^\beta$. The perturbed fields transform according to $\delta u^\alpha = T^\alpha{}_\mu(\delta^\mu{}_\beta + D^\mu{}_\beta)\delta u_0^\beta$. (The extra bit $D^\mu{}_\beta$ comes from the metric dependence in $T^\alpha{}_\beta$.
- It is straightforward to show that the following transformation laws are satisfied:

$$\begin{aligned} \delta E &= \delta E_0 + S^0_{\mu\nu} D^\mu{}_\alpha (2\delta^\nu{}_\beta + D^\nu{}_\beta)\delta u_0^\alpha \delta u_0^\beta, \\ \bar{C}_{\alpha\beta}\delta u^\alpha \delta u^\beta &= \left[\bar{C}^0_{\alpha\beta} + 2S^0_{\mu\nu} D^\mu{}_\alpha (2\delta^\nu{}_\sigma + D^\nu{}_\sigma) F_0^\sigma{}_\beta + E_{\alpha\beta}\right]\delta u_0^\alpha \delta u_0^\beta. \end{aligned}$$

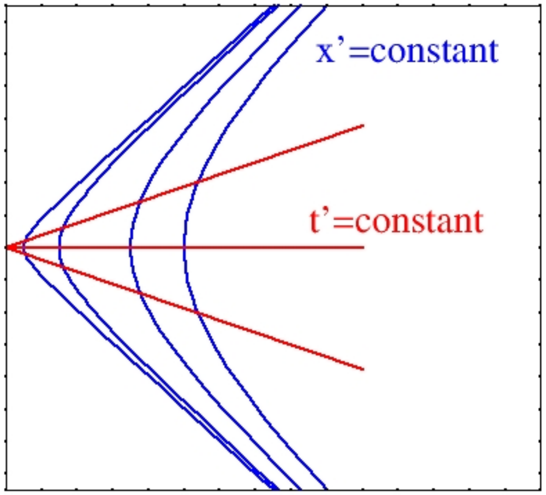
- In the Rindler geometry $D^\mu{}_\nu = 0$, so the instability growth times are independent of the kinematical parameters.
- In the Painlevé-Gullstrand form of the Schwarzschild metric $D^\mu{}_\nu$ depends only on $\hat{z}$, so none of the growth timescales in this geometry can depend on any of the other kinematical parameters.

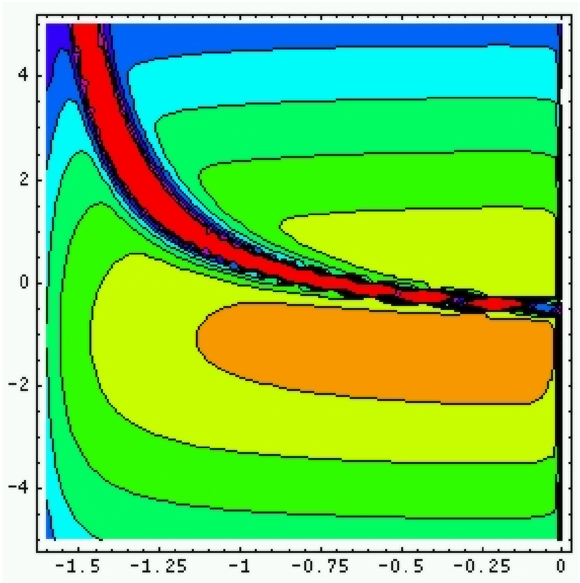
- We have tested these ideas by studying the perturbations of Rindler space:

$$ds^2 = -x'^2 dt'^2 + dx'^2 + dy^2 + dz^2.$$

- It can be shown that $1/\tau$ in Rindler can depend only on the two parameters (γ, ζ) and this has been (partially) verified numerically.

- The eigenvalues of $\bar{C}_{\alpha\beta}$ can be determined analytically for Rindler as a function of (γ, ζ):

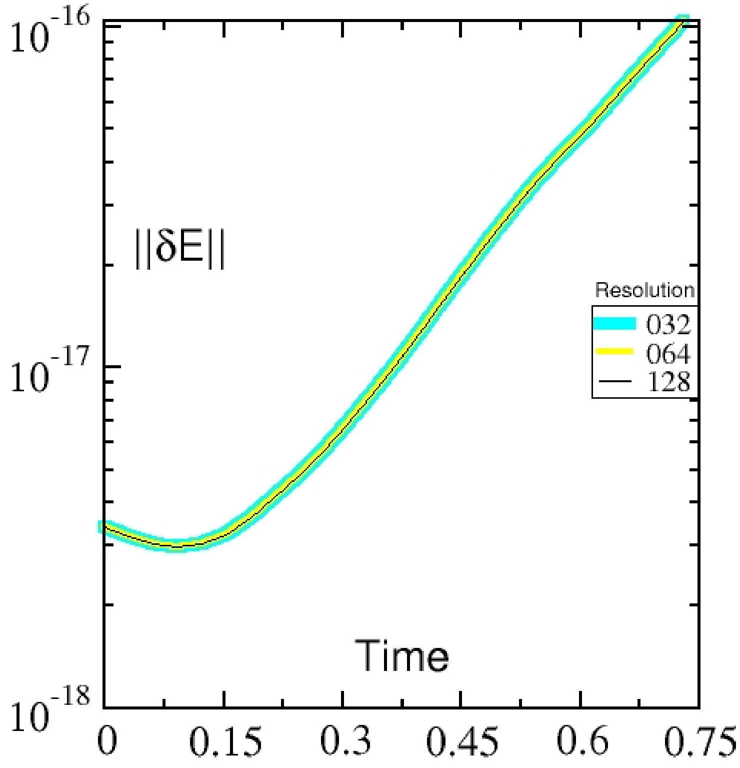




Declaring our Oranges

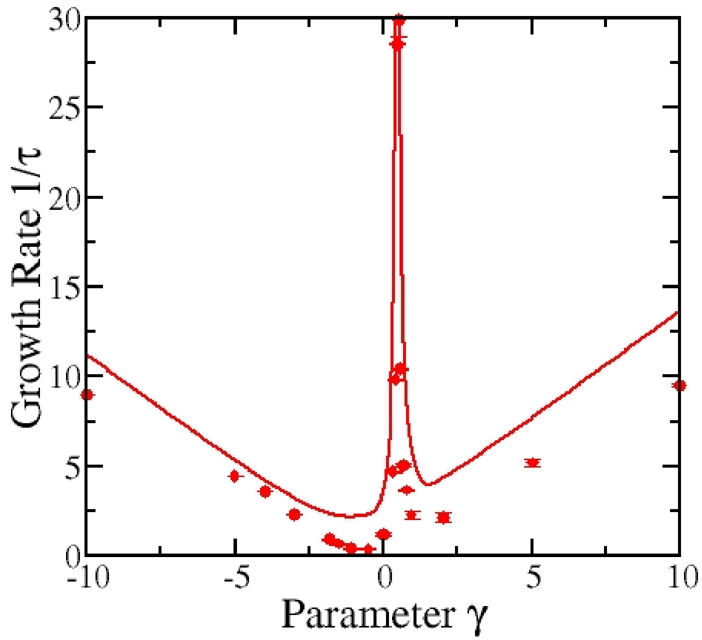
- Numerical computations were (are being) done with a linearized version of the Cornell code (KPST).
- Many results presented here were computed using both pseudo-spectral and finite difference versions of the code.
- All results presented here were run at several spatial resolutions with convergence obtained in the final results.
- Initial data were taken to be (small) Gaussian pulses in all dynamical fields.

- Typical evolution of the energy $||\delta E|| = \sqrt{\int \delta E \; d^3x}$ in Rindler.



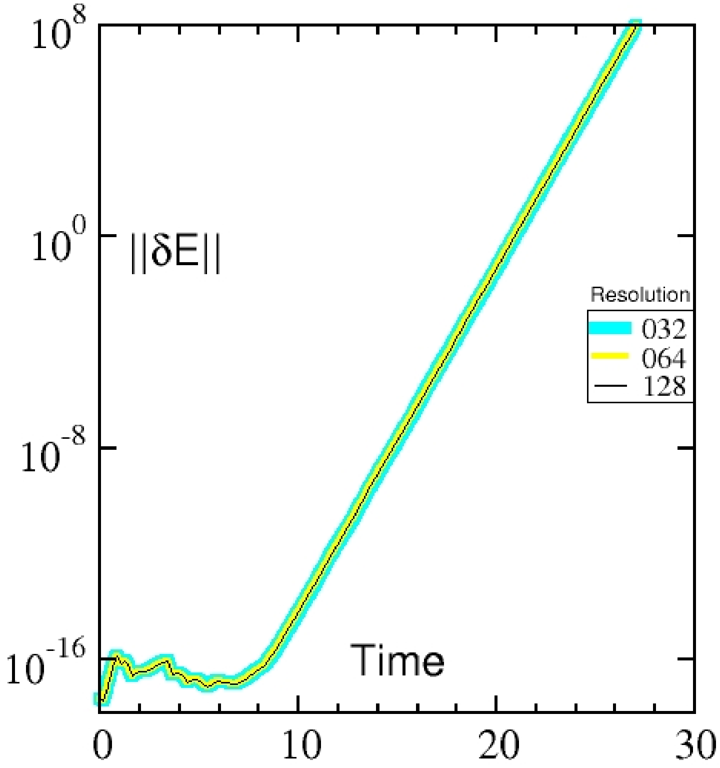
- Graphs show results computed with three pseudo-spectral resolutions having 32, 64, and 128 basis functions in the x' direction.

- The growth timescales $1/\tau$ obtained numerically (points) are compared to the analytical estimates obtained from the eigenvalues of $\bar{C}_{\alpha\beta}$ for a range of the KST parameter γ.

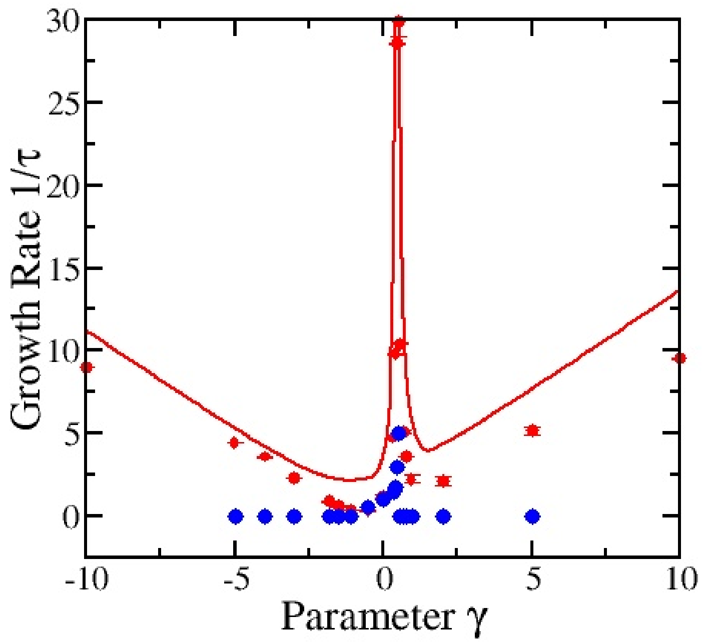


- The estimates have good qualitative agreement with the numerical results.

- On longer timescales the solution stabilizes and then may show a subsequent instability that grows at a slower rate.



- Timescales for this long-term instability are shown as blue points:



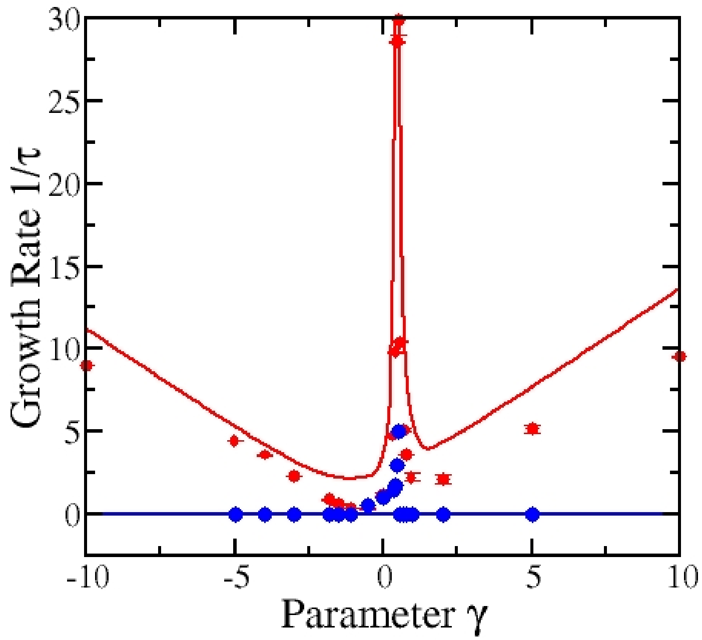
- Long-term instability timescales do not agree well with the estimates based on the largest eigenvalue of $\bar{C}_{\alpha\beta}$. For most values of the parameter γ the evolutions appear stable !

- Short-term instability in Rindler is caused by an unstable in- or out-going mode. After (roughly) the light crossing time of the computational domain, these modes leave and the remaining solution is (more) stable.

- Long-term instability in Rindler (when it occurs) is caused by a non-propagating mode.

- New improved growth timescale estimate: project $\bar{C}_{\alpha\beta}$ into the sub-space of non-propagating modes (i.e. eigenvectors of $A^k_{\alpha\beta}$ with vanishing characteristic speeds) before finding its eigenvalues.

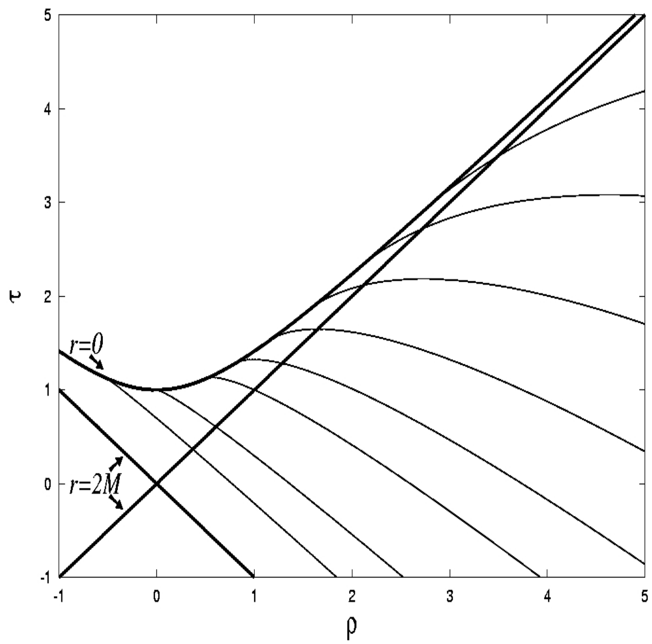
- Solid blue curve gives the analytical non-propagating mode estimate for the Rindler growth timescale.



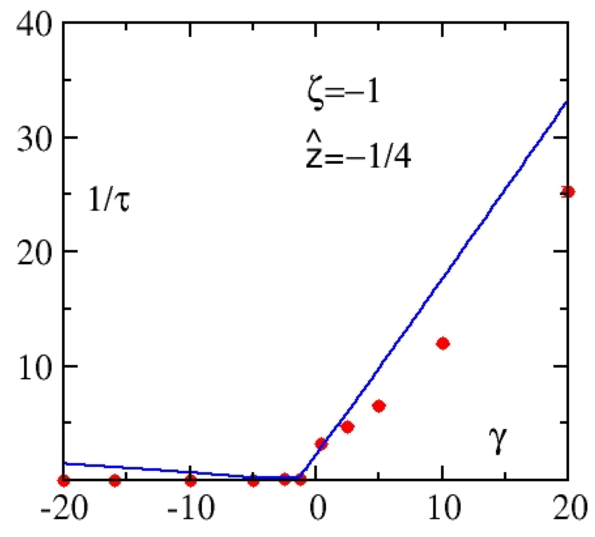
- Except for points where the propagating mode growth rate is very large, this estimate correctly predicts long-term stability.

- Next consider the stability of the evolution equations in a neighborhood of the Schwarzschild geometry.
- We use the Painlevé-Gullstrand form of the Schwarzschild metric:

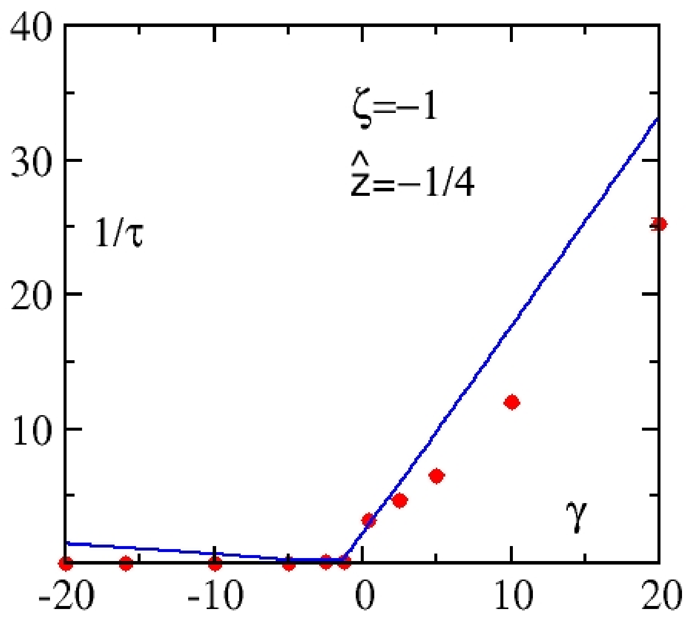
$$ds^2 = -dT^2 + (dr + \sqrt{2M/r}dT)^2 + r^2 d\Omega^2.$$

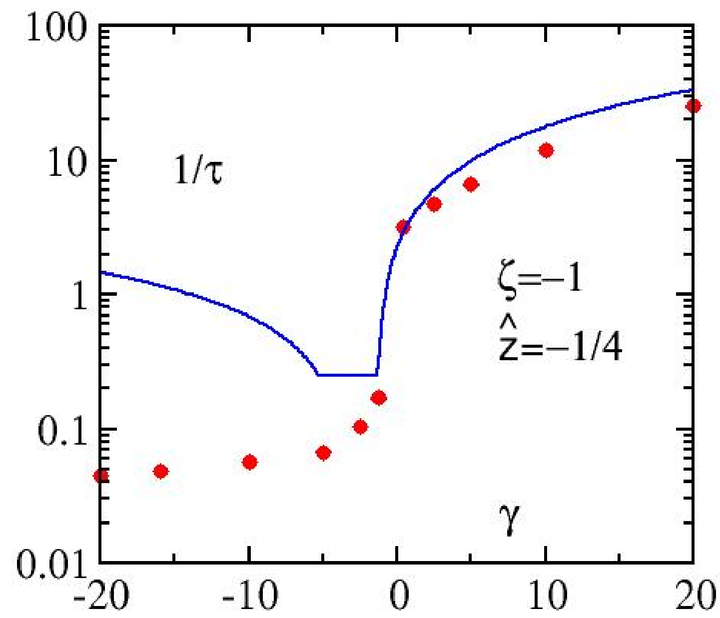


- In PG coordinates the in-going and non-propagating modes of Schwarzschild cross the horizon and leave the computational grid on a timescale comparable to M.
- To estimate the long-term growth rate of Schwarzschild (in PG coordinates) project $\bar{C}_{\alpha\beta}$ onto the sub-space of out-going modes (i.e. the eigenvectors of $A^k_{\alpha\beta}$ with eigenvalue $v = +1$) before finding its eigenvalues.



- These (very preliminary) results for Schwarzschild are promising.
- It is also clear that our crude growth-rate estimates fail to capture some critical feature(s) of Schwarzschild stability.
- It remains to be seen whether these estimates can be used to find the most stable representation of the equations for the case of a binary black hole spacetime.





Evidence (albeit indirect) suggests Einstein preferred Oranges to Apples:

A.M. Prep

If you work the morning shift you'll need to make sure you have everything ready for the day so that things run smoothly when the lunch rush starts. Some things to have ready:

- At least 5 bins of tomatoes. 2 on the line, and 3 underneath.
- At least 2 small containers of avocado mash.
- A bowl of cream cheese and at least one large container of Vinagarette out to begin warming up to room temperature.
- The soup signs made for the day.
- 2 bowls of oranges sliced.
- The water containers filled with ice and water (and one with lemon and lime slices).
- Coffee made and milk for the coffee in a thermos.
- All the bins on the line full.
- The cafe clean, chairs down, produce put away, and the sign placed outside.

Review

What city did the idea for Einstein's come from?
Berkley
Holland
San Francisco
Reno

Did You Know?

A squeeze of lime or lemon juice, a dash of olive oil, and a shake of salt makes the avocado mash taste better and stay green longer.

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