BSSN: why/when does it work?

Miguel Alcubierre

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The linearized ADM equations

Consider the vacuum ADM equations, with geodesic slicing and zero shift. Consider also a linear perturbation of flat space:

$$g_{ij} = \delta_{ij} + h_{ij} \qquad h_{ij} << 1$$

The evolution equations become:

$$\partial_t h_{ij} = -2K_{ij}, \qquad \partial_t K_{ij} = -1/2 \left(\nabla^2_{flat}\, h_{ij} - \partial_i \Gamma_j - \partial_j \Gamma_i \right)$$

$$\Gamma_i \equiv \sum_k \partial_k h_{ik} - 1/2\, \partial_i trh$$

And the constraints are:

$$\sum_k \partial_k f_k = 0 \quad \text{(energy)} \;, \qquad \partial_t f_i = 0 \quad \text{(momentum)}$$

$$f_i \equiv \sum_k \partial_k h_{ki} - \partial_i trh = \Gamma_i - 1/2\, \partial_i trh$$

Fourier analysis of linear ADM

Consider a Fourier mode: $h_{ij} = \hat{h}_{ij}\, e^{i(\omega t - kx)}, \quad K_{ij} = \hat{K}_{ij}\, e^{i(\omega t - kx)}$

Define now: $\vec{h} = (\hat{h}_{xx}, \hat{h}_{yy}, \hat{h}_{zz}, \hat{h}_{xy}, \hat{h}_{xz}, \hat{h}_{yz})$

The evolution equations imply:

$$\omega^2 \vec{h} = k^2 M \vec{h},$$

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The constraints become:

$$\hat{h}_{yy} + \hat{h}_{zz} = 0 \qquad (\text{ham, mom}_x)$$

$$\hat{h}_{xy} = 0 \qquad (\text{mom}_y)$$

$$\hat{h}_{xz} = 0 \qquad (\text{mom}_z)$$

Eigenstructure of ADM

The eigenvalues and eigenvectors of the matrix *M* are:

- $\lambda=0$ (non-propagating), with eigenvectors
  - $v_1=(1,0,0,0,0,0)$ gauge (satisfies all constraints)
  - $v_2=(0,0,0,1,0,0)$ violates mom-y
  - $v_3=(0,0,0,0,1,0)$ violates mom-z
- $\lambda=1$ (speed of light), with eigenvectors
  - $v_4=(2,1,1,0,0,0)$ violates ham + mom-x
  - $v_5=(0,1,-1,0,0,0)$ transverse-traceless
  - $v_6=(0,0,0,0,0,1)$ transverse-traceless

The linearized BSSN equations

Again consider linear perturbation of flat space. The evolution equations of BSSN become (m=constant):

$$\partial_t \phi = -K/6, \quad \partial_t \tilde{h}_{ij} = -2\tilde{A}_{ij}$$

$$\partial_t K = 0,$$

$$\partial_t \tilde{A}_{ij} = -1/2\left(\nabla^2_{flat}\tilde{h}_{ij} - \partial_i \tilde{\Gamma}_j - \partial_j \tilde{\Gamma}_i\right) - 2\left(\partial_i \partial_j \phi + \delta_{ij}\nabla^2_{flat}\phi\right)$$

$$\partial_t \tilde{\Gamma}_i = -2\sum_k \partial_k \tilde{A}_{ki} + m\left(2\sum_k \partial_k \tilde{A}_{ki} - 4/3\,\partial_i K\right)$$

And the constraints:

$$\sum_k \partial_k \tilde{f}_k = 0 \quad \text{(energy)}, \qquad \partial_t \tilde{f}_i = 0 \quad \text{(momentum)}$$

$$\tilde{f}_i \equiv \sum_k \partial_k \tilde{h}_{ki} - 8\partial_i \phi$$

Eigenstructure of BSSN

Consider again a Fourier mode and take: $\tilde{\vec{h}} = (\phi, \tilde{\hat{h}}_{xx}, \tilde{\hat{h}}_{yy}, \tilde{\hat{h}}_{zz}, \tilde{\hat{h}}_{xy}, \tilde{\hat{h}}_{xz}, \tilde{\hat{h}}_{yz})$

The eigenvalues and eigenvectors are:

- λ=0 (non-propagating), with eigenvectors
  - v_1=(1,8,-4,-4,0,0,0) gauge (satisfies all constraints)
- λ=m, with eigenvectors
  - v_2=(0,0,0,0,1,0,0) violates mom-y
  - v_3=(0,0,0,0,0,1,0) violates mom-z
- λ=1 (speed of light), with eigenvectors
  - v_4=(0,0,1-m,1-m,0,0,0) violates trh=0
  - v_5=(0,0,1,-1,0,0,0) transverse-traceless
  - v_6=(0,0,0,0,0,0,1) transverse-traceless
- λ=2m-1, with eigenvectors
  - v_7=(0,1,0,0,0,0,0) violates ham + mom-x + trh=0

When is linearized BSSN stable

If we want all eigenvalues to be positive (and hence all characteristic speeds to be real) we must have:

- $m > 0$
- $2m-1 > 0 \quad \Rightarrow \quad m > 1/2$

The second condition in fact contains the first one.

Notice that *m* measures how much momentum constraint is added to the evolution equation for the gammas, so we must add some!

Notice also that for *m* positive, all constraint violating modes propagate.

The natural choice is $m=1$, in which case all propagating modes have the speed of light (this is the standard BSSN choice).

Conclusions

- Non-propagating modes can in fact grow linearly even in the linearized case. In a nin-linear case, non-propagating modes can easily blow up.
- ADM has 3 constraint violating modes, two of which are non-propagating.
- BSSN has 4 constraint violating modes, all propagate with the speed of light.
- BSSN only *requires* that one adds the momentum constraints to the evolution equation for the gammas in order to get real characteristic speeds for the constraint violating modes.

Examples: Flat space in non-trivial coordinates

- Example 1, trivial slicing, non-trivial spatial coordinates:

$$r_{Minkowski} = r'\left(1 - a\, f(r')\right)$$

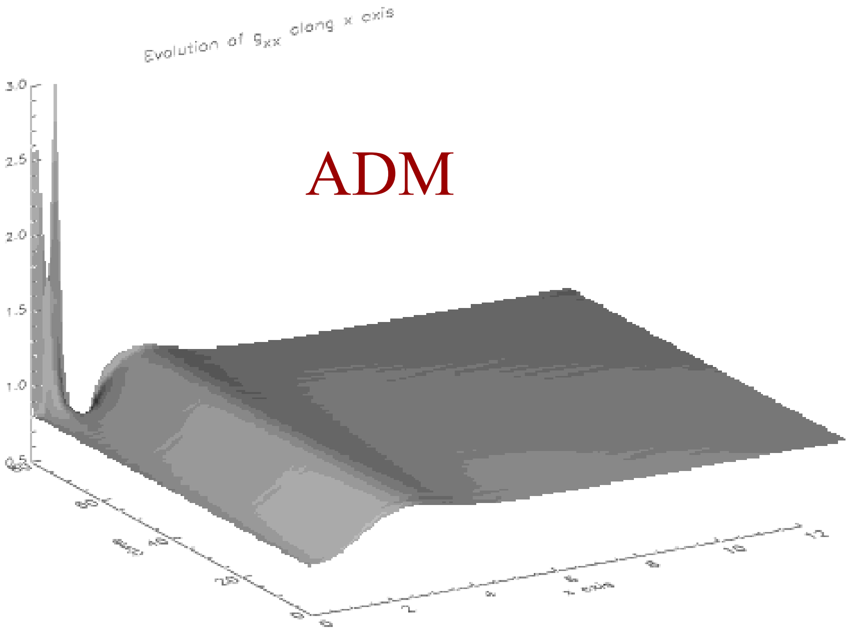
with f(r´) a gaussian centered at r=0 of amplitude 1 and 0<a<1.

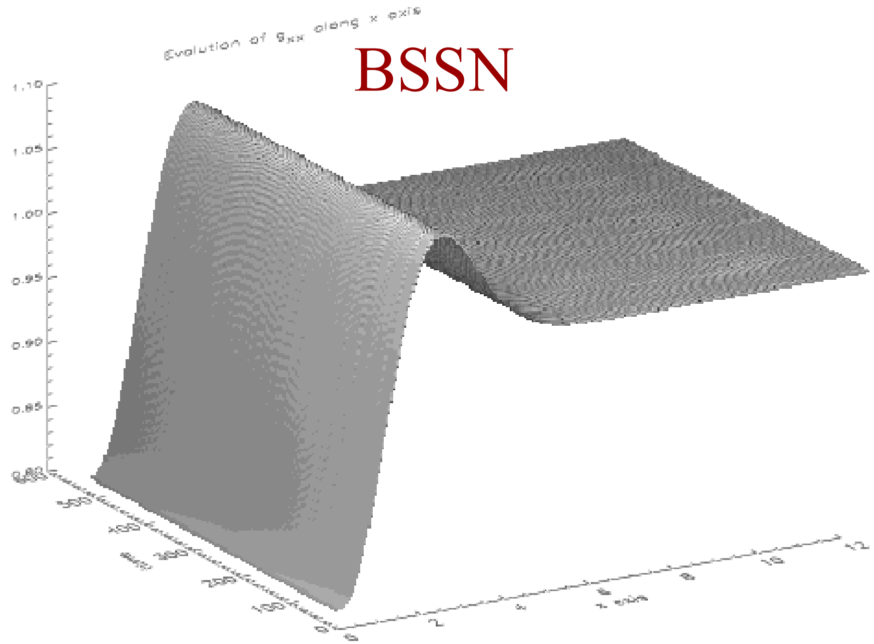
- Example 2, non-trivial slicing:

$$t' = t - a\, f(r)$$

if -1<a df/dr<1 this results in spatial slices.

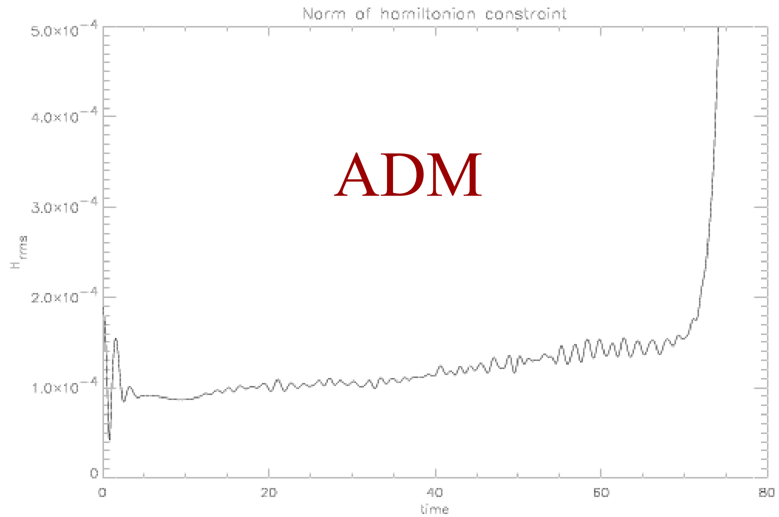
BSSN vs. ADM: flat space in non-trivial spatial coordinates

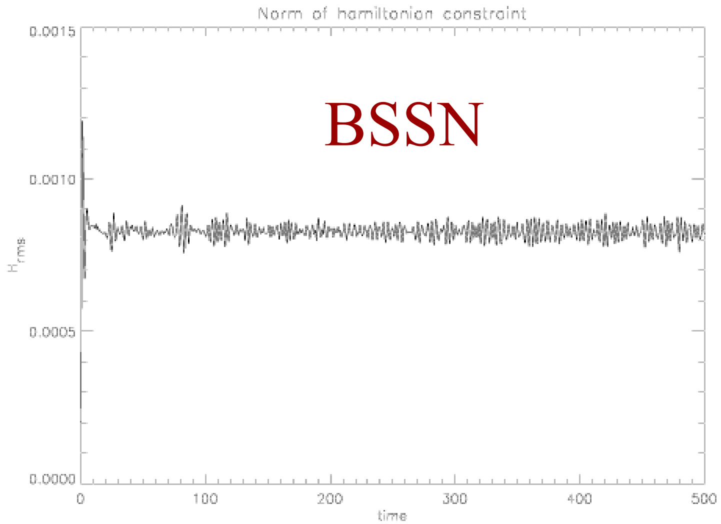




Evolution of g_{xx} along x-axis

BSSN vs. ADM: flat space in non-trivial spatial coordinates

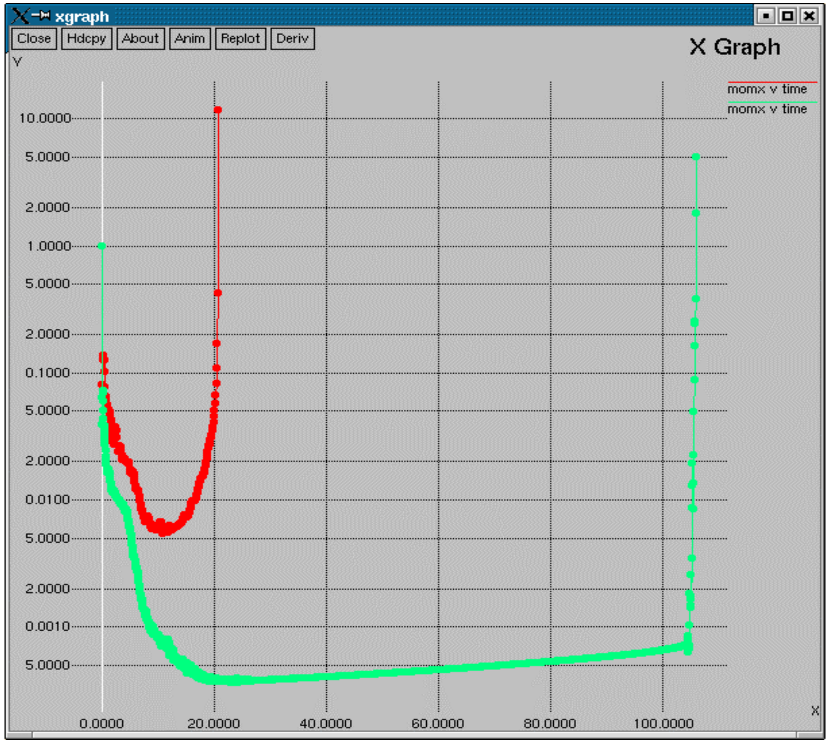




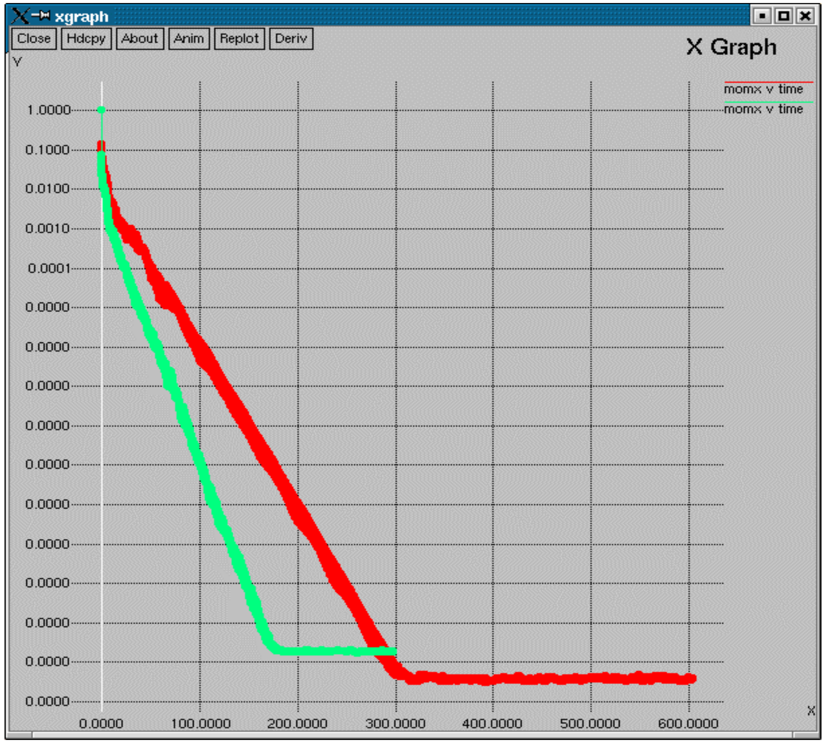
Norm of the hamiltonian constraint

BSSN: flat space in non-trivial spatial coordinates: one octant

Static lapse, two resolutions

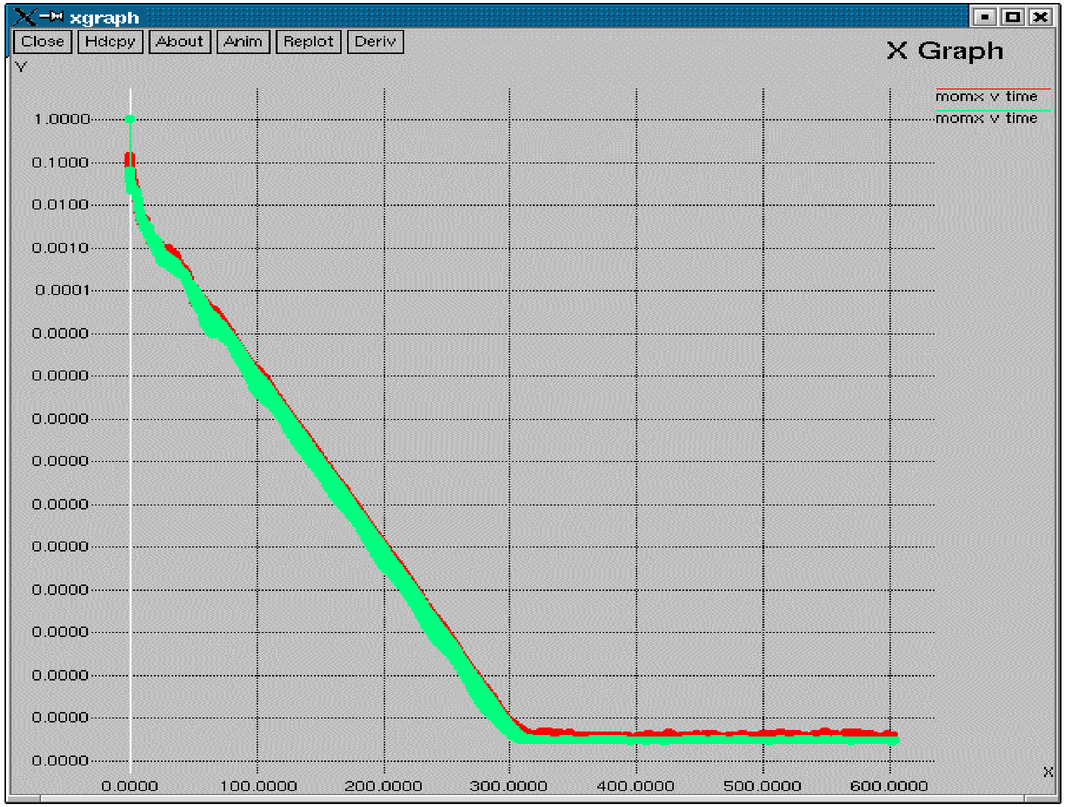


Harmonic slicing, two resolutions



Norm of momentum constraint

BSSN: flat space in non-trivial spatial coordinates: octant vs. full



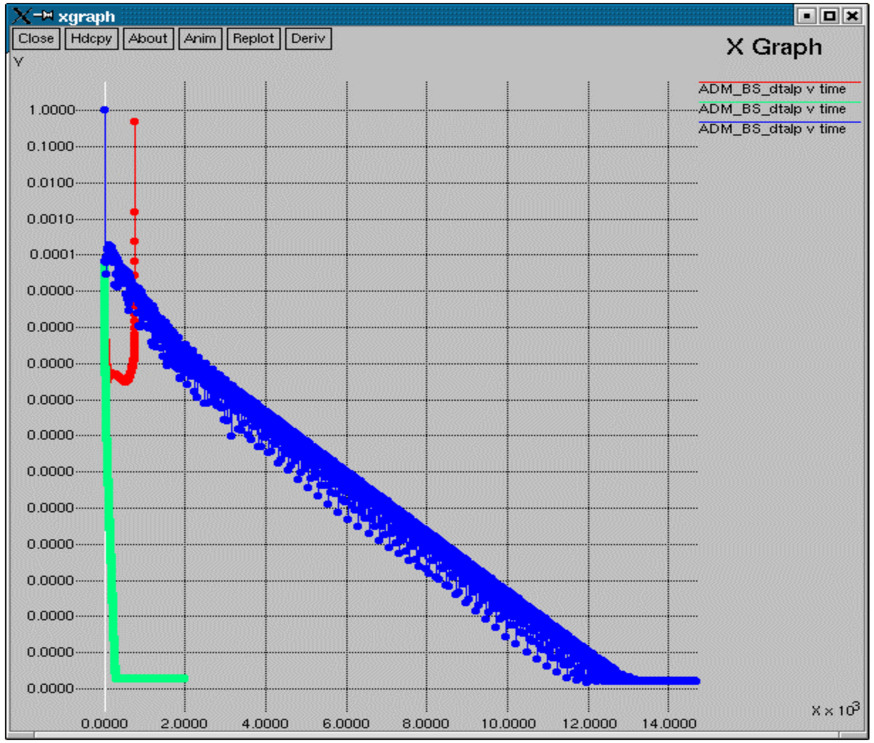
Norm of momentum constraint, octant vs. full

(harmonic slicing)

BSSN: flat space in non-trivial slicing: one octant

- Red line: full BSSN, harmonic slicing, static shift.
- Green line: frozen Gammas, harmonic slicing, static shift
- Blue line: full BSSN, harmonic slicing, hyperbolic shift.

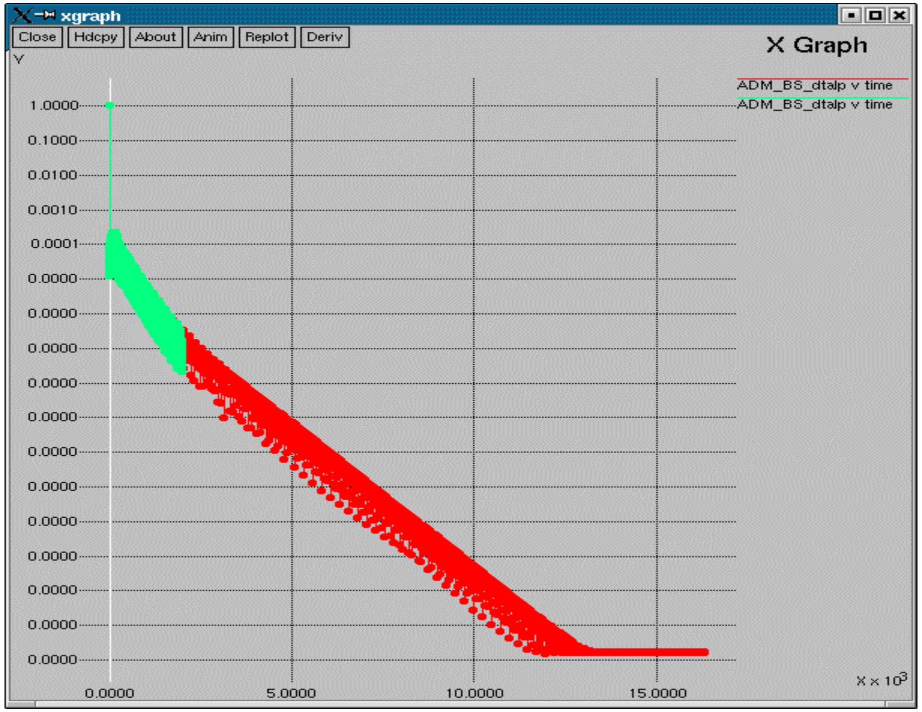
Norm of change in lapse



BSSN: flat space in non-trivial slicing: octant vs. full

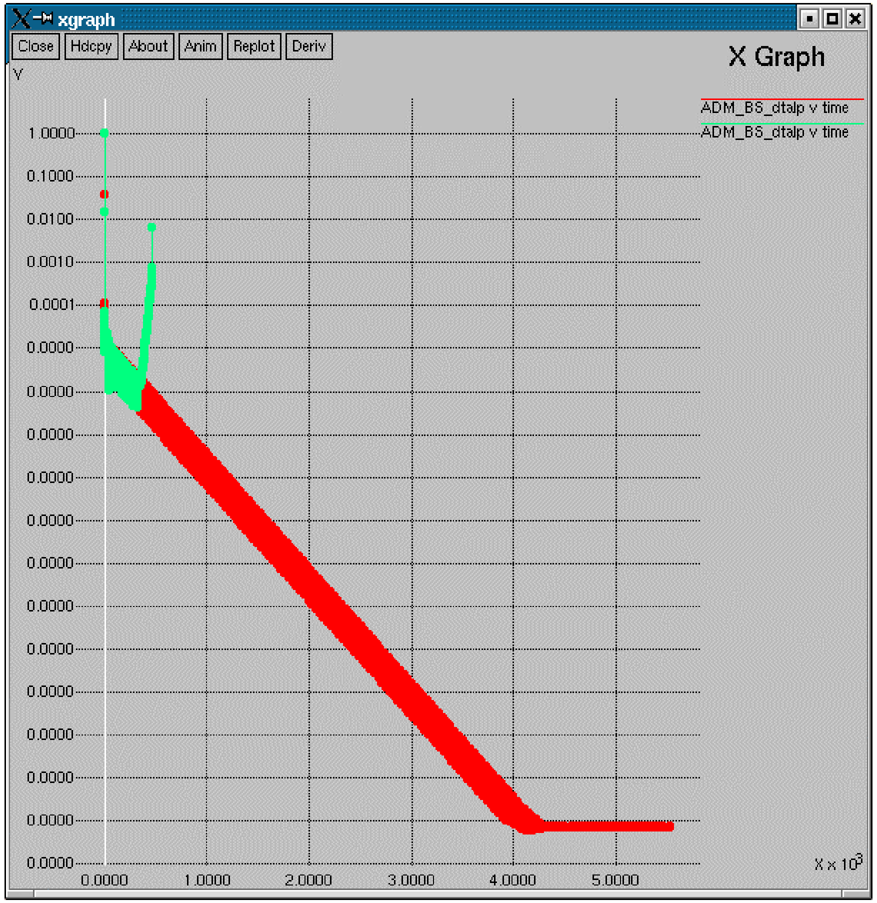
Full BSSN, harmonic slicing, hyperbolic shift.

Norm of change in lapse



BSSN: Excised Kerr-Schild non-rotating BH: octant vs. full

Norm of change in lapse

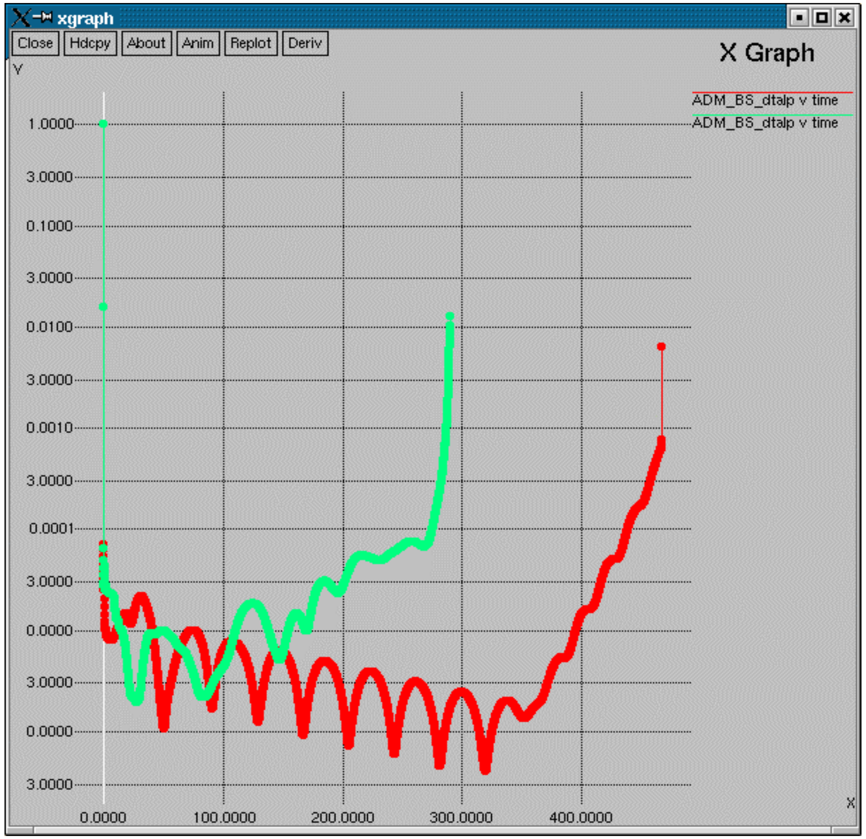


Full BSSN, harmonic slicing, static shift.

- Red line: 1 octant
- Green line: full grid

BSSN: Excised Kerr-Schild non-rotating BH: octant vs. full

Norm of change in lapse



Full BSSN, harmonic slicing, full grid.

- Green line: static shift
- Red line: hyperbolic shift