Gauge stability of 3+1 formulation of general relativity

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Analytical Instabilities

- Physical Instability
  – Caustic, singularity
- Constraint instability
  – Constraints violation
- Gauge instability

Gauge Perturbation: method of mathematical analysis

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

– Infinitesimal coordinate transformation

$$x^\mu \to x^\mu + \xi^\mu$$

$$g_{\mu\nu} \to g_{\mu\upsilon} - (\nabla_\mu \xi_\nu + \nabla_\mu \xi_\nu)$$

$$U \equiv \delta\alpha\,, \quad V_i \equiv \delta\beta_i \quad W_{ij} \equiv \delta\gamma_{ij}$$

$$\delta g_{00} = -2\alpha U + 2\gamma^{ij}\beta_i V_j - \gamma^{kj}\beta_i \beta_j W_{jk}$$

$$\delta g_{0i} = V_i$$

$$\delta g_{ij} = W_{ij}$$

$$\frac{\partial \xi_0}{\partial t} = \alpha U - \beta^i V_i - \beta^i \beta^j \frac{\partial \xi_i}{\partial x^j} + (\Gamma^\nu_{00} + \beta^i \beta^j \Gamma^\nu_{ij}) \xi_\nu$$

$$\frac{\partial \xi_i}{\partial t} = -V_i - \frac{\partial \xi_i}{\partial x^j} + 2\,\Gamma^\nu_{0i} \xi_\nu$$

- *During the integration …*

$$F_\mu(x^\nu, \alpha, \frac{\partial \alpha}{\partial x^\nu}, \ldots, \beta_i, \frac{\partial \beta_i}{\partial x^\nu}, \ldots, \gamma_{ij} \frac{\partial \gamma_{ij}}{\partial x^\nu}, \ldots) = 0$$

$$\Rightarrow \frac{\partial F_\mu}{\partial \alpha} U + \frac{\partial F_\mu}{\partial (\frac{\partial \alpha}{\partial x^\nu})\, \partial x^\nu} + \ldots + \frac{\partial F_\mu}{\partial \beta_i} V_i + \ldots = 0$$

Eight Quasi-linear partial different equations

$$\vec{z}(x^{\mu}) = (\xi^{\mu}, U, V_i)^T$$

$$\frac{\partial z_r}{\partial t} = {}^{(0)}M_{\mu\nu} z_\nu + {}^{(1)}M^{i}_{\mu\nu} \frac{\partial z_\nu}{\partial x^i} + {}^{(2)}M^{ij}_{\mu\nu} \frac{\partial^2 z_\nu}{\partial x^i \partial x^j} \cdots$$

Short wavelength perturbation …:linear analysis , roots of the characteristic polynomial

$$\vec{z} = \vec{\zeta}(t) \exp(-iqe_k x^k)$$

$$\frac{\partial \vec{\zeta}}{\partial t} = \hat{M}(t)\zeta$$

$$\hat{M}(t, e_i, q) = {}^{(0)}M_{\mu\nu} + {}^{(1)}M^{i}_{\mu\nu}(t) i e_i q + {}^{(2)}M^{ij}_{\mu\nu}(t) e_i e_j q^2 + \ldots$$

- Neglect time dependence of $\hat{M}$

$$\vec{z}(x^{\mu}) \propto \exp\left(\omega_s t - iq e_k x^k\right)$$

- Wavenumber

$$\omega_s(q, e_i) \Rightarrow \det(\hat{M} - \omega \hat{N}) = 0$$

- $Re(\omega_s)$ can be used to probe ill-posedness

$$\because \mathrm{Re}(\omega) \to \infty \ \left(q \to \infty\right)$$

➔Determine the growth of instabilities with in time.

Gauge instability

- Fixed gauge (. e.g. synchronous gauge …):
  - $a = a(x^\mu)$, $\beta_i = \beta_i(x^\mu)$.
- Algebraic gauge (.e.g.1+log slicing …):
  - $a = a(x^\mu, g_{ij})$, $\beta_i = \beta_i(x^\mu)$.
- Differential gauge (.e.g.Maximal slicing):
  - $\gamma^{ij}\nabla_i\nabla_j\alpha = K^{ij}K_{ij}\alpha\ ,\quad \beta_i = 0$

Fixed Gauge

$$\alpha=\alpha(x^{\mu}),\beta_i=\beta_i(x^{\mu})$$

$$\Downarrow$$

Follow the gauge perturbation procedures.

$$\Downarrow$$

Harmonic solutions with $\mathrm{Re}(\omega) \sim q^{1/2}$
→ Ill-posed

Synchronous gauge: $\alpha=\alpha(t)$, $\beta=0$ can avoid this

$$\because \quad \omega \sim O(1)$$

Algebraic Gauge

$$\alpha=\alpha(x^{\mu},\gamma_{ij})\ ,\beta_i=\beta_i(x^{\mu})$$

$$\Downarrow$$

Follow the same procedures as the above

$$\left(\beta^i\beta^j+\frac{\partial\alpha^2}{\partial\gamma_{ij}}\right)e_ie_j>0,\quad\forall e_i$$

Well posed !

However, stability depend on choice of shift!

future work ....

$$\alpha=\alpha(x^{\mu},\gamma_{ij})\ ,\quad\beta_i=\beta_i(x^{\mu},\gamma_{ij})$$

Note

$$\left(\beta^i \beta^j + \frac{\partial \alpha^2}{\partial \gamma_{ij}}\right) e_i e_j > 0, \quad \forall e_i$$

$$\Downarrow$$

$$\frac{\partial \alpha}{\partial t} - \beta^i \frac{\partial \alpha}{\partial x^i} = -\alpha^2 f(\alpha) tr(K_{ij}), \quad f > 0$$

$$\log(\alpha \gamma^{-\sigma}) = tr(Q^\mu), \quad \beta_i = \beta_i(x^\mu), \quad \sigma > 0$$

Differential Gauge: Maximal slicing gauge

$$\gamma^{ij}\nabla_i\nabla_j\alpha = K^{ij}K_{ij}\alpha, \ \beta_i = 0$$

$$\frac{\partial\alpha}{\partial t} = \frac{1}{\varepsilon}(\gamma^{ij}\nabla_i\nabla_j\alpha - K^{ij}K_{ij}\alpha), \ \beta_i = 0, \varepsilon > 0$$

$$\Downarrow$$

Through the same procedure

$\mathrm{Re}(\omega) \propto q^{1/2}$ → Ill-posed

Note: ill-posedness comes from asymptotic $q^{1/2}$ behavior of parabolic structure

$$\because \frac{\partial^2\xi_0}{\partial t^2} = \Gamma^i_{00}\frac{\partial\xi_0}{\partial x^i} + \ldots$$

Physical meaning of gauge instability

- ADM 3+1 formulation

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i$$

$$\partial_t K_{ij} = -\nabla_i \nabla_j \alpha + \beta^k \nabla_k K_{ij} + \left(K_{ik} \nabla_j + K_{jk} \nabla_i\right) \beta^\kappa + \alpha \left(R^{(3)}{}_{ij} + K_{ij} \mathrm{tr} K - 2 K_{ik} K^k{}_j\right)$$

-Flat spacetime: One-dimensional solutions

$$\gamma_{ij} = diag(\gamma,1,1), \quad K_{ij} = diag(K,0,0)$$

$$\Downarrow$$

$$\frac{\partial u}{\partial t} = A(u) + \hat{B}(u)\frac{\partial u}{\partial x}$$

For β is non-zero →
strong hyperbolic and well posed

stability analysis $\Downarrow$

Well posed: $\mathrm{Re}(\omega) \sim O(1)$

However, unstable for small perturbation unless

$$\left(\frac{\partial \beta}{\partial x} = \frac{\partial \gamma}{\partial x} = \frac{\partial \alpha}{\partial x} = 0, \quad K > 0 \right)$$

$$\because \frac{\partial \alpha}{\partial x} \neq 0, \quad \mathrm{Re}(\omega) \sim \beta^{-1} \to \infty \ (\beta \to 0)$$

- Instability of a synchronous coordinate system

Synchronous gauge has caustic (blow up) instability

However $\Downarrow$

This is different instability from respect to small perturbations of coordinates

-Gauge instabilities in acceleration systems:

Using the Rindler frame which is constructed from the world lines of particles moving with accelerations

Ill-posedness of fixed gauges with spatially non-uniform lapse $\Longrightarrow$ deformation of the frame and the prescribed acceleration of particles

Ill-posed to well posed but unstable gauge with $\mathrm{Re}(\omega)\alpha\beta^{-1}$ $\Longrightarrow$ Fixed coordinate will decrease with increasing β

- Instability of rotating reference frames

$$\alpha = 1, \quad \beta_i = \{-\Omega x^2, \Omega x^1, 0\}, \quad \gamma_{ij} = diag(1,1,1)$$

Rotation and Coriolis forces do not change the nature of instability

Summary

- Fixed gauge is ill-posed except synchronous gauge
- Algebraic gauge is well-posed and stable depending on choice of gauge.
- Differential gauge is ill-posed without shift.

Gauge instabilities have explicit physical meaning …

Future Works

- Extend stability analysis to particular type of gauge: .e.g. 1+log slicing, hyperbolic shifts … .e.g. …

$$\partial_t \alpha = -\alpha^2 f(\alpha)(K - K_0)/\Psi^4,$$

$$\partial_t^2 \beta^i = \frac{k}{\Psi^n} \partial_t \tilde{\Gamma}^i - \eta \partial_t \beta^i$$

- Numerical experiments!