# How to make ADM-like versions of the Einstein equations well-posed

Carsten Gundlach, U Southampton

José María Martín-García, CSIC

- Well-posedness for first-order systems eg Gustafsson, Kreiss & Oliger
- Constraint-preserving boundary conditions Calabrese et al (LSU group)
- Well-posedness for second-order systems Kreiss & Ortiz, Nagy, Ortiz & Reula
- Constraint-preserving boundary conditions
- Making BSSN well-posed

# FIRST ORDER IN SPACE AND TIME

## Well-posedness

Start from a system of quasilinear, first-order evolution equations:

$$\partial_t u = P^i(u)\,\partial_i u + S(u)$$

where $u(x^i, t)$ is a vector, $P^i$ square matrices.

The problem is well-posed if the solution exists, is unique, and depends continuously on the initial data.

# Linearization

Linearize around a background solution $u_0$:

$$\partial_t \delta u = P^i(u_0)\, \partial_i \delta u + Q(u_0) \delta u$$

Well-posedness requires that the solution of the linearized problem is bounded as

$$||\delta u(t)|| \leq f(t) ||\delta u(0)||$$

where $f(t)$ *does not depend on* $\delta u(0)$.

## Frozen coefficients approximation

We are interested in potential high-frequency instabilities: approximate $P^i(u_0)$ and $Q(u_0)$ as constant.

$Q$ contributes a factor $e^{|Q|t}$ to the bound $f(t)$: neglect it.

This leaves

$$\partial_t \delta u = P^i \partial_i \delta u$$

## Fourier transform

Writing $u$ instead of $\delta u$ now,

$$\partial_t u = P^i \, \partial_i u$$

Fourier transform

$$u(x,t) = \hat{u}(t) e^{i\omega_i x^i}$$

gives an ODE system

$$\partial_t \hat{u} = i|\omega| P_n u$$

where

$$P_n \equiv n_i P^i, \qquad n_i \equiv \frac{\omega_i}{|\omega|}$$

## Strong hyperbolicity

If $P_n$ can be diagonalized as $P_n = T\Lambda T^{-1}$ with $\Lambda$ real and $T$ bounded in $n_i$, then

$$|\widehat{u}(t)| \leq K|\widehat{u}(0)|$$

for all $\omega_i$, and so

$$||u(t)|| \leq K||u(0)||$$

With a complex eigenvalue $a \pm ib$,

$$|\widehat{u}(t)| \sim e^{b|\omega|t}$$

With a Jacobi block of size $k$,

$$|\widehat{u}(t)| \sim (|\omega|t)^{k-1}$$

In either case the blow-up depends on $|\omega|$ and hence on $u(x, 0)$.

In numerics, $|\omega|$ is set by the spatial resolution.

## Characteristic variables

Fix a unit vector $n^i$. Decompose $\partial_i = (\partial_n, \partial_A)$.

$U$ is a characteristic variable with respect to $n_i$ with speed $\lambda$ if

$$\partial_t U = \lambda \partial_n U + \partial_A \dots$$

For a first order system

$$U = T^{-1}$$

where $T$ is the matrix of eigenvectors of $P_n$:

$$P_n T = T \Lambda$$

## Maximally dissipative boundary conditions

Bound on positive definite energy $\Rightarrow$ Bound on $||u(t)|| \Rightarrow$ Well-posedness

$$E = \frac{1}{2}\int_V u^\dagger H u \, dV$$

where $H$ is positive definite, independent of $n_i$, and $HP^i$ is symmetric $\Leftrightarrow H = (T^{-1})^\dagger B T^{-1}$

$$\begin{aligned} \frac{dE}{dt} &= \int_V \partial_i (u^\dagger H P^i u) \, dV \\ &= \int_{\partial V} (u^\dagger H P_n u) \, dS \\ &= \int_{\partial V} (U^\dagger B \Lambda U) \, dS \\ &\sim \int_{\partial V} (U_+^2 - U_-^2) \, dS \end{aligned}$$

Make $dE/dt \le 0$ by imposing the boundary conditions

$$U_+ = b U_- \quad \text{where } |b| \le 1$$

## Constraint-preserving boundary conditions

Closed system of quasilinear constraints with an energy.

For each $C_\pm$ there is a $U_\pm$ with

$$C_\pm = \partial_n U_\pm + \partial_A \ldots$$

Maximally dissipative boundary conditions:

$$C_+ = bC_- \quad \text{where } |b| \leq 1$$

Defining $X \equiv U_+ - bU_-$ gives

$$\partial_n X = 0$$

Write as an evolution equation for $X$ on the boundary:

$$\partial_t X = \partial_A \ldots + \ldots$$

Impose the *compatible* boundary condition

$$U_+ = bU_- + X$$

on the main system. Calabrese et al (LSU group)

# FIRST ORDER IN TIME, SECOND ORDER IN SPACE

## Pseudo-differential reduction to first order

Wave equation written first order in time, second order in space:

$$\begin{aligned}\partial_t \phi &= \pi \\ \partial_t \pi &= \partial_i \partial^i \phi\end{aligned}$$

Fourier transform:

$$\begin{aligned}\partial_t \hat{\phi} &= \hat{\pi} \\ \partial_t \hat{\pi} &= -|\omega|^2 \hat{\phi}\end{aligned}$$

Spread powers of $|\omega|$ around by $\hat{\varphi} \equiv i|\omega|\hat{\phi}$:

$$\begin{aligned}\partial_t \hat{\varphi} &= i|\omega|\hat{\pi} \\ \partial_t \hat{\pi} &= i|\omega|\hat{\varphi}\end{aligned}$$

"First order in space" and strongly hyperbolic.

Kreiss & Ortiz

## Characteristic variables for a second order in space, first order in time system

Characteristic variables (in the direction $n_i$) can be defined as before by

$$\partial_t U = \lambda \partial_n U + \partial_A(\ldots)$$

where now $U$ is made from $\partial_i u$.

By this definition any transversal derivative is a zero speed characteristic variable:

$$\partial_t(\partial_A u) = \partial_A(\ldots)$$

Non-zero speed characteristic variables are unique only up to addition of transversal derivatives:

$$U \to U + \partial_A(\ldots)$$

## Maximally dissipative boundary conditions

We need an energy

$$E = \int_V \text{pos.def.}\,(u, \partial_i u)\, dV$$

conserved in the sense that

$$\frac{dE}{dt} \sim \int_{\partial V} (U_+^2 - U_-^2)\, dS \qquad (*)$$

1. Use pseudo-differential reduction to first order to find the $U_\pm$ (with non-zero speeds).

2. Add suitable transversal derivatives $\partial_A(\ldots)$ terms to obtain (*).

3. Impose boundary conditions

$$U_+ = bU_- + \text{given} \quad \text{where } |b| \le 1$$

where $U_\pm$ are characteristic with respect to the normal on the boundary.

4. Constraint-preserving BCs as before.

Gundlach & Martin-Garcia

## ADM form of the Einstein equations

3+1 split of the spacetime metric

$$ds^2 = -\alpha^2\, dt^2 + \gamma_{ij}(dx^i + \beta^i\, dt)(dx^j + \beta^j\, dt)$$

Shorthand notation

$$\partial_0 \equiv \alpha^{-1}(\partial_t - \mathcal{L}_\beta)$$

Evolution equations

$$\begin{aligned} \partial_0 \gamma_{ij} &= -2K_{ij} \\ \partial_0 K_{ij} &= -\alpha^{-1} D_i D_j \alpha + R_{ij} - 2K_{il}K^{lj} + KK_{ij} \end{aligned}$$

Constraints

$$\begin{aligned} H &\equiv R - K_{ij}K^{ij} + K^2 = 0 \\ M_i &\equiv D_j K^j{}_i - D_i K = 0 \end{aligned}$$

# NOR modifications of ADM

Principal part of evolution equations is

$$\begin{aligned}\partial_0\gamma_{ij} &\simeq -2K_{ij}\\ \partial_0 K_{ij} &\simeq \frac{1}{2}\gamma^{kl}\left(-\gamma_{ij,kl}-\gamma_{kl,ij}+\gamma_{ki,jl}+\gamma_{kj,il}\right)\end{aligned}$$

1. Densitize the lapse:

$$\alpha = (\det\gamma)^{\sigma/2}\,Q$$

with $\sigma > 0$ and $Q(x^i, t)$ given instead of $\alpha$.

2. Define auxiliary variables

$$f_i \equiv \gamma^{jk}\gamma_{ij,k}$$

This gives

$$\partial_0 K_{ij} \simeq \frac{1}{2}\gamma^{kl}\left(-\gamma_{ij,kl}-(1+\sigma)\gamma_{kl,ij}+f_{i,j}+f_{j,i}\right)$$

3. Use the momentum constraint to evolve $f_i$:

$$\partial_t f_i \simeq -2\gamma^{jk}K_{ij,k} \simeq -2K_{,i}$$

# Results

- The NOR system has a strongly hyperbolic pseudo-differential reduction to first order, and so is well-posed. (Nagy, Ortiz and Reula 2003).

- BSSN with a densitized lapse, and the algebraic constraints $trK = 0$ and $\det \gamma = 1$ imposed, is also well-posed in this sense.

- NOR and BSSN can be given conserved, positive definite energies for the main system and the constraint system, for physical propagation speeds.

- NOR admits a family of maximally dissipative boundary conditions which are compatible with the constraints.

- We think this will work in BSSN too.