Suggestions for simple tests of boundary conditions

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December 2003

Or....

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Or.... A collection of obvious generalities plus things that don't work yet.

Tests for 3D numerical relativity

The tests outlined at the first Mexico workshop and paper were

- simple physical systems
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They were also incomplete. Many things were not discussed. One extremely important topic is boundary conditions.

Motivation

There are two main questions I would like to pose:

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- What's the simplest thing we can get away with?

What's a good boundary condition?

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6. Degree of *physical* understanding.

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Robust stability can be performed on "asymmetric" grids (strips) to reduce overhead, boundaries can cut irregularly into the grid, etc.

BSSN is robustly stable even with boundaries

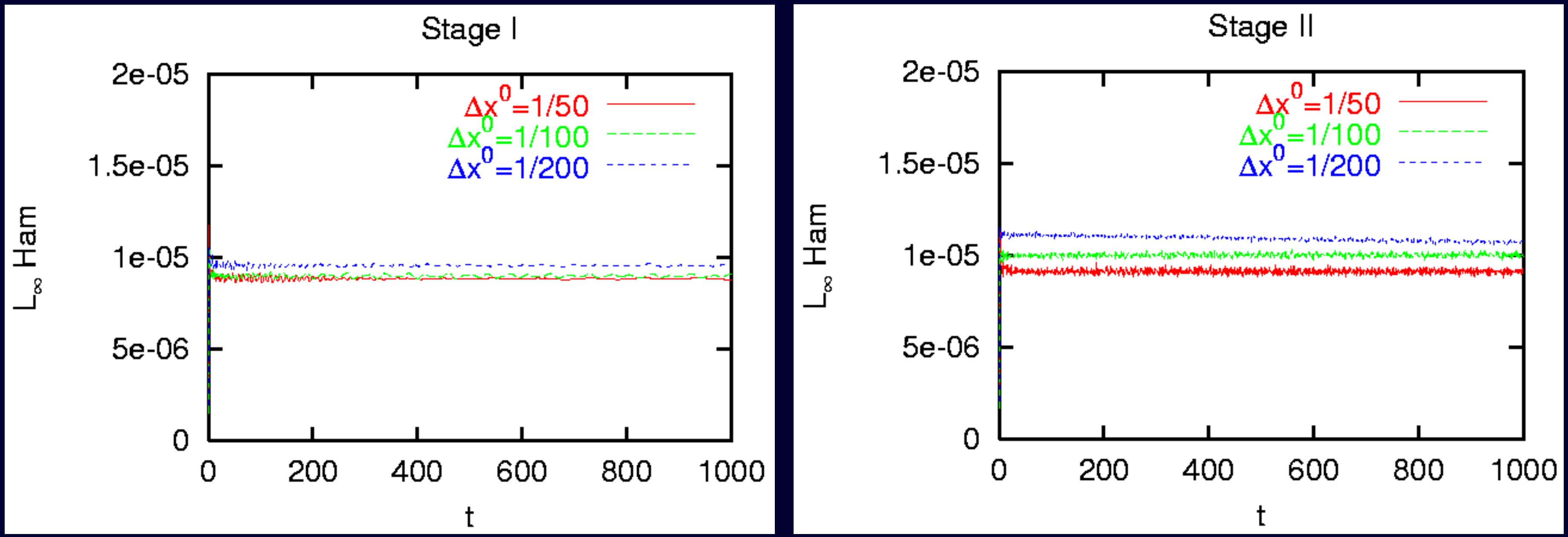

Figure 1: Robust stability test (stages I and II) for BSSN.

Figure from Schnetter, Hawley, Hawke (gr-qc/0310042)

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Instead, I propose to use a very boring spacetime.

A modest proposal

1. Set up Minkowski space in a non-standard coordinate system

2. Evolve, extracting the physical information in

   (a) the regime causally disconnected to the boundary, checking that this converges to the Minkowski space answer
   (b) the regime causally connected to the boundary, finding the effects through the convergence test.

Multiple initial coordinate systems, boundary locations, etc. will be needed to look at the effects.

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In standard coordinates Minkowski space has

$$\Theta = \frac{1}{2r}.$$

For the purposes of tests a surface of constant Θ can be treated like a horizon.

Analyzing the results

Then we need to analyze the results of the finder on the CE surface. To look at the effect on the mass of an AH I suggest looking at the area of the CE surface. In the case where the CE surface is an AH this is related to the irreducible mass of the BH.

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To look at the effect on the spin of the hole I suggest using the isolated horizon formalism. A numerical implementation was described in Dreyer et al., 2003. Here I use the code written by Erik Schnetter. The isolated horizon formalism requires the existence of a (local, approximate) Killing vector so the spacetime must be (locally and approximately) axisymmetric.

Simple test of physically extracted quantities

- Initially Minkowski space with standard Cauchy data $\gamma_{ij} = \delta_{ij}, \quad K_{ij} = 0$.
- Lapse initially $\alpha = 1$, evolved with harmonic slicing.
- Shift initially a constant rigid rotation, held static. Rotation speed $\Omega = 1$ "superluminal" in grid corners.
- $x^i \in [-1, 1]$.
- Find CE surface, $\Theta = 4$ (at $r = 0.5$).
- As a first try I used standard BSSN with Sommerfeld type boundary conditions.

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- The area and coordinate radius of the surface is practically unchanged to output precision!

- The spin measured on the surface does change, but in a resolution dependent manner. This may be due to the ill-posed boundary conditions applied.

Spin depends on resolution

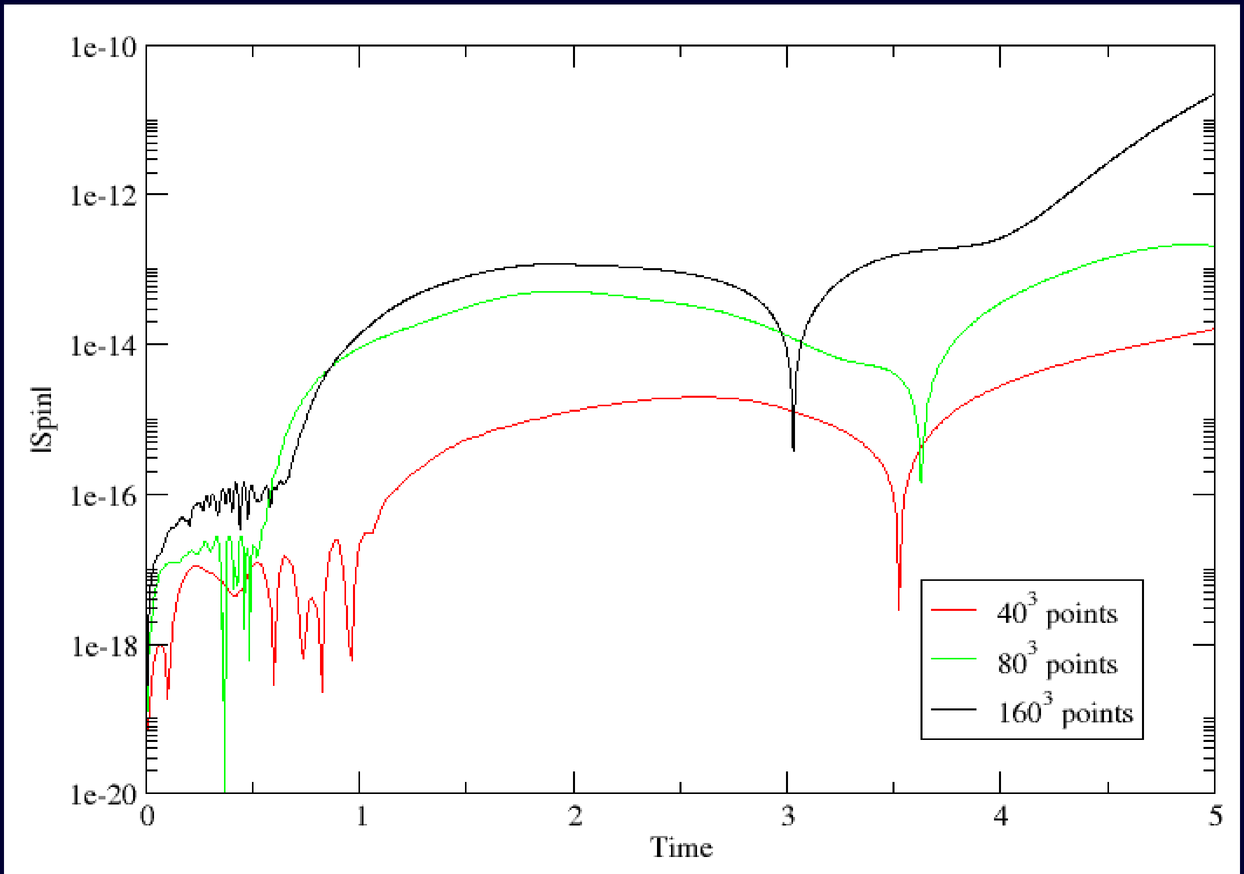

Figure 2: The spin depends on the resolution. It is most noticeable that increasing the resolution makes the spin increase. This may be due to the ill-posed boundary conditions applied.

Conclusions

These suggestions for tests are clearly inadequate as they stand. Particular things spring to mind:

- The robust stability test is only picking up weakly hyperbolic formulations. It looks like it doesn't show the loss of well-posedness for BSSN with Sommerfeld BCs on all variables.
- The errors induced in these tests are so small that the results are slightly ambiguous. Also, this is in contrast to "real physics" runs. Some way of inducing larger errors just at the boundaries would be good.
- That the "mass" doesn't change may not be surprising with a constant rotation. A mass scale will probably be needed to test this effect.