

# Absorbing Boundary Conditions

Apples with Apples, December 2 2003

Mihaela Chirvasa, Horst Beyer, Denis Pollney

# Overview

- **artificial boundary:** conditions at the edges of the computational domain in such a way that the solution in interior is practically unaffected by the presence of the boundary.

# Overview

- **artificial boundary:** conditions at the edges of the computational domain in such a way that the solution in interior is practically unaffected by the presence of the boundary.
- In general it is not possible to prescribe perfect absorbing boundary conditions unless the solution is known in the region of infinity (Gustafsson & Kreiss, 1979).

# Overview

- **artificial boundary:** conditions at the edges of the computational domain in such a way that the solution in interior is practically unaffected by the presence of the boundary.
- In general it is not possible to prescribe perfect absorbing boundary conditions unless the solution is known in the region of infinity (Gustafsson & Kreiss, 1979).
- Exact boundary conditions for general problems, if they are found, are usually *nonlocal* and *global in time* so they are not useful for numerical purposes. Practically we have to find *the best local absorbing boundary conditions*.

## Overview

- **artificial boundary:** conditions at the edges of the computational domain in such a way that the solution in interior is practically unaffected by the presence of the boundary.
- In general it is not possible to prescribe perfect absorbing boundary conditions unless the solution is known in the region of infinity (Gustafsson & Kreiss, 1979).
- Exact boundary conditions for general problems, if they are found, are usually *nonlocal* and *global in time* so they are not useful for numerical purposes. Practically we have to find *the best local absorbing boundary conditions*.
- Require that data has *compact support* in the computational domain.

# Overview

- **artificial boundary:** conditions at the edges of the computational domain in such a way that the solution in interior is practically unaffected by the presence of the boundary.
- In general it is not possible to prescribe perfect absorbing boundary conditions unless the solution is known in the region of infinity (Gustafsson & Kreiss, 1979).
- Exact boundary conditions for general problems, if they are found, are usually *nonlocal* and *global in time* so they are not useful for numerical purposes. Practically we have to find *the best local absorbing boundary conditions*.
- Require that data has *compact support* in the computational domain.
- For general evolution systems these boundaries should accomplish:
  - ★ *outgoing modes* should go out without producing spurious reflections
  - ★ *ingoing modes* should not be allowed to enter in the grid
  - ★ *tangential modes* should remain in the grid.

## Sommerfeld Boundary Condition

Consider the wave equation in 4-dimensional Minkowski space for the field  $u$ :

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} . \quad (1)$$

and *spherically symmetric* initial data with compact support in the computational domain. Then at the edge of the grid the wave will be purely outgoing, of the form:

$$u = \frac{f(r - t)}{r} \quad (2)$$

Then the *exact* boundary condition for this system will be the Sommerfeld BC:

$$(\partial_r + \partial_t)(ru)|_{boundary} = 0 \quad (3)$$

- This boundary condition is appropriate for radiation problems, for which the waveforms will approach spherical symmetry, about the centre of the system at large distances.
- If the data is not spherical symmetric then we will have strong reflections at the boundary.
- Need better boundary conditions...

## Absorbing Boundary Conditions for scalar wave equation (the approach of Engquist/Majda 1977)

Consider the wave equation in 3-dimensional Minkowski space for the field  $u$ :

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \quad (4)$$

Assume  $u$  to correspond to *data having support in the half-plane  $x \geq 0$* .

**Goal:** Find the artificial boundary which serves to suppress the reflection of outgoing 'modes' at  $x = 0$

- consider plane wave solutions travelling to the left

$$u = e^{i\omega \left( \sqrt{1 - \frac{k_y^2}{\omega^2}} x + t \right)} e^{ik_y y}, \quad (5)$$

where  $k_y \in \mathbb{R}$ ,  $|\omega| > |k_y|$

- they satisfy in particular

$$\left[ \left( \frac{\partial}{\partial x} - i\omega \sqrt{1 - \frac{k_y^2}{\omega^2}} \right) u \right] \Big|_{x=0} = 0 \quad (6)$$

- substitute in this relation

$$\omega \text{ with } \frac{1}{i} \frac{\partial}{\partial t} \text{ and } k_y \text{ with } \frac{1}{i} \frac{\partial}{\partial y} \quad (7)$$

- result: boundary condition which is *non local* in space and in time.
- perform Taylor expansion

$$\sqrt{1 - \frac{k_y^2}{\omega^2}} \approx 1 - \frac{k_y^2}{2\omega^2} - \frac{k_y^4}{8\omega^4} + \dots \quad (8)$$

- sequence of local boundary conditions (first, second, and third order approximations):

$$\left( \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} \right) \Big|_{x=0} = 0 ,$$

$$\left( \frac{\partial^2 u}{\partial t \partial x} - \frac{\partial^2 u}{\partial t^2} + \frac{1}{2} \frac{\partial^2 u}{\partial y^2} \right) \Big|_{x=0} = 0 ,$$

$$\left( \frac{\partial^4 u}{\partial t^3 \partial x} - \frac{\partial^4 u}{\partial t^4} + \frac{1}{2} \frac{\partial^4 u}{\partial t^2 \partial y^2} + \frac{1}{8} \frac{\partial^4 u}{\partial y^4} \right) \Big|_{x=0} = 0 , \text{ etc.}$$

(9)

- Using (4) the previous equations can be put in the form

$$\begin{aligned} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right) u \Big|_{x=0} &= 0, \\ \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right)^2 u \Big|_{x=0} &= 0, \\ \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right)^2 \left[ \frac{\partial^2}{\partial t^2} - \frac{1}{4} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right)^2 \right] u \Big|_{x=0} &= 0, \text{ etc.} \end{aligned} \tag{10}$$

- Results

- ★ well-posedness:

1. By an *energy estimate* they show the first order boundary condition leads to a well-posed problem
2. Using a theorem of Kreiss they conclude that the second order boundary condition also leads to a well-posed problem
3. The third order is ill-posed

- ★ Reflection of the outgoing modes

1. normal incidence: no reflection
2. at  $45^\circ$ : reflection coefficients are: 0.17, 0.029, 0.0055
3. close to glancing: all the reflection coefficients tend to unity.

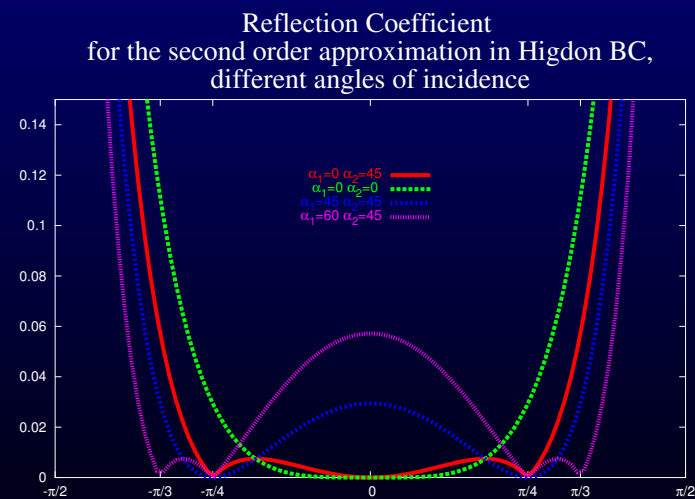
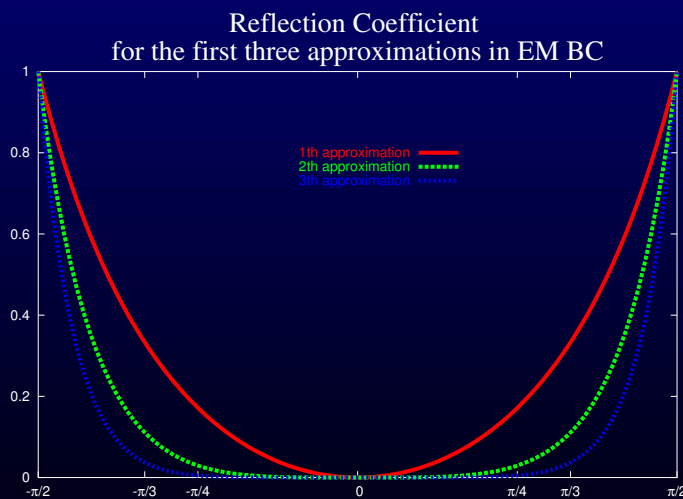
- Generalization

A more *general form* of the EM boundary conditions was found by Higdon(1986) (appropriate for dispersive problems )

$$\left[ \prod_{j=1}^p \left( (\cos \alpha_j) \frac{\partial}{\partial t} - c_j \frac{\partial}{\partial x} \right) \right] u = 0 \quad (11)$$

- reflection coefficient (for a plane wave traveling with wave speed  $c$  and hitting the  $x$ -boundary at angle of incidence  $\cos \theta$ )

$$R = \prod_{j=1}^p \left| \frac{c \cos \alpha_j - c_j \cos \theta}{c \cos \alpha_j + c_j \cos \theta} \right| \quad (12)$$



# Numerical Tests of Absorbing Boundary Conditions for Scalar Wave Equation

## Evolution equations and initial data:

$$\begin{aligned} \left( \partial_{tt} - \Delta + m^2 \right) u &= 0 \\ u(x, y, z, 0) &= f(x, y, z) \\ \partial_t u(x, y, z, 0) &= g(x, y, z) \end{aligned}$$

implementation: leap-frog scheme:

$$\begin{aligned} \phi_{ijk}^{n+1} = & 2\phi_{ijk}^n \left( 1 - \rho_x^2 - \rho_y^2 - \rho_z^2 - \frac{m^2(\Delta t)^2}{2} \right) - \rho_x^2 \left( \phi_{i+1jk}^n - \phi_{i-1jk}^n \right) \\ & - \rho_y^2 \left( \phi_{ij+1k}^n - \phi_{ij-1k}^n \right) - \rho_z^2 \left( \phi_{ijk+1}^n - \phi_{ijk-1}^n \right) - \phi_{ijk}^{n-1} \end{aligned}$$

$(\rho_x, \rho_y, \rho_z)$ -Courant factors)

# Absorbing Boundary Conditions

$$(\partial_x - \cos \alpha \partial_t)|_{x\text{-boundary}} u = 0 \quad (\text{Abs I})$$

$$(\partial_x - \cos \alpha_1 \partial_t) (\partial_x - \cos \alpha_2 \partial_t)|_{x\text{-boundary}} u = 0 \quad (\text{Abs II})$$

## Absorbing Boundary Conditions

$$(\partial_x - \cos \alpha \partial_t)|_{x\text{-boundary}} u = 0 \quad (\text{Abs I})$$

$$(\partial_x - \cos \alpha_1 \partial_t) (\partial_x - \cos \alpha_2 \partial_t)|_{x\text{-boundary}} u = 0 \quad (\text{Abs II})$$

- The relations (Abs I) and (Abs II) are the 1st and 2nd order absorbing BC for a *plane wave* hitting the boundary with the angle  $\alpha$  in the case (Abs I) and with the angle  $\alpha_1$  or  $\alpha_2$  for (Abs II).

## Absorbing Boundary Conditions

$$(\partial_x - \cos \alpha \partial_t)|_{x\text{-boundary}} u = 0 \quad (\text{Abs I})$$

$$(\partial_x - \cos \alpha_1 \partial_t) (\partial_x - \cos \alpha_2 \partial_t)|_{x\text{-boundary}} u = 0 \quad (\text{Abs II})$$

- The relations (Abs I) and (Abs II) are the 1st and 2nd order absorbing BC for a *plane wave* hitting the boundary with the angle  $\alpha$  in the case (Abs I) and with the angle  $\alpha_1$  or  $\alpha_2$  for (Abs II).
- Putting  $\alpha = \alpha_1 = \alpha_2 = 0$  we get the boundary conditions derived by Endquist&Majda in '77

## Absorbing Boundary Conditions

$$(\partial_x - \cos \alpha \partial_t)|_{x\text{-boundary}} u = 0 \quad (\text{Abs I})$$

$$(\partial_x - \cos \alpha_1 \partial_t) (\partial_x - \cos \alpha_2 \partial_t)|_{x\text{-boundary}} u = 0 \quad (\text{Abs II})$$

- The relations (Abs I) and (Abs II) are the 1st and 2nd order absorbing BC for a *plane wave* hitting the boundary with the angle  $\alpha$  in the case (Abs I) and with the angle  $\alpha_1$  or  $\alpha_2$  for (Abs II).
- Putting  $\alpha = \alpha_1 = \alpha_2 = 0$  we get the boundary conditions derived by Endquist&Majda in '77
- We compare these with the radiative boundary conditions:

$$(\partial_r + \partial_t) (ru)|_{\text{boundary}} = 0 \quad (\text{Rad I})$$

$$(\partial_r + \partial_t)^2 (ru)|_{\text{boundary}} = 0 \quad (\text{Rad II})$$

## Absorbing Boundary Conditions

$$(\partial_x - \cos \alpha \partial_t)|_{x\text{-boundary}} u = 0 \quad (\text{Abs I})$$

$$(\partial_x - \cos \alpha_1 \partial_t) (\partial_x - \cos \alpha_2 \partial_t)|_{x\text{-boundary}} u = 0 \quad (\text{Abs II})$$

- The relations (Abs I) and (Abs II) are the 1st and 2nd order absorbing BC for a *plane wave* hitting the boundary with the angle  $\alpha$  in the case (Abs I) and with the angle  $\alpha_1$  or  $\alpha_2$  for (Abs II).
- Putting  $\alpha = \alpha_1 = \alpha_2 = 0$  we get the boundary conditions derived by Endquist&Majda in '77
- We compare these with the radiative boundary conditions:

$$(\partial_r + \partial_t) (ru)|_{\text{boundary}} = 0 \quad (\text{Rad I})$$

$$(\partial_r + \partial_t)^2 (ru)|_{\text{boundary}} = 0 \quad (\text{Rad II})$$

- The relations (Rad I) and (Rad II) are the perfect absorbing BC for a *spherical wave* propagating radially. The relation (Rad I) is the usual Sommerfeld condition.

## Global quantities used to classify the BC

- **Energy**

$$E(t) = \int \int \int \left( \left( \frac{\partial u}{\partial t} \right)^2 + |\nabla u|^2 + m^2 u^2 \right) dV$$

- **Norms of the reflected wave**

- ★ Infinity norm:

$$\|\phi\|_{\infty} = \max |\phi_{ijk} - \phi_{ijk}^{exact}|$$

- ★ L2-norm:

$$\|\phi\|_2 = \sqrt{\sum |\phi_{ijk} - \phi_{ijk}^{exact}|^2 / N}$$

- ★ Sobolev Norm:

$$\|\phi\|_s = \sum \left( |\nabla (\phi_{ijk} - \phi_{ijk}^{exact})|^2 + m^2 (\phi_{ijk} - \phi_{ijk}^{exact})^2 \right) / N$$

where  $\phi_{ijk}$  is a grid variable and  $\phi_{ijk}^{exact}$  is its exact value.

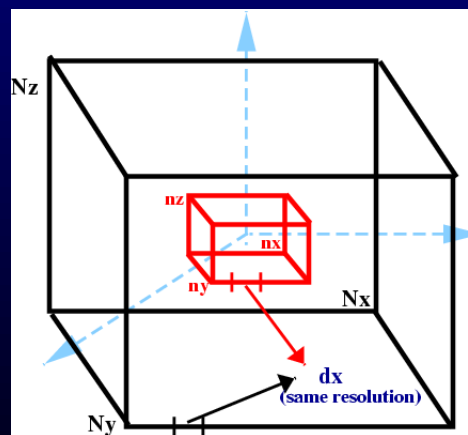
## Calculation of $\phi_{ijk}^{exact}$

In order to obtain a measure of the reflected wave, we perform tests on two grids: A small grid, and a much larger grid which contains the small grid.

We perform the runs for as long as the small grid is causally disconnected from the large grid boundaries.

For this period of time, the solution on the large grid is regarded as the exact solution, uncontaminated by boundary effects:  $\phi_{ijk}^{exact}$

We can compare the solution on the small grid to  $\phi_{ijk}^{exact}$  to determine the quality of boundary conditions.



## Results

We are interested to see how these boundary conditions behave for different types of initial data:

1. **spherically symmetric pulse, centered at the origin ID**
  - gaussian
2. **spherically symmetric pulse, and off-centered ID**
  - gaussian off-centered
3. **nonspherically symmetric pulse ID**
  - gaussian\*ReY22

Grid parameters:

$$\left\{ \begin{array}{ll} nx = ny = nz = 81 & \textit{small grid} \\ Nx = Ny = Nz = 641 & \textit{big grid} \end{array} \right.$$

$$dx = dy = dz = 0.0005 \quad \rho_x = 0.5; \quad dt = 0.00025 \quad (\sigma = 0.005)$$

-- >run until  $t = 4$  crossing times =  $4 \times 160$  iterations = 0.16

(the perturbation from the boundary of the big grid appears at  $t = 0.14$ )

# RESULTS

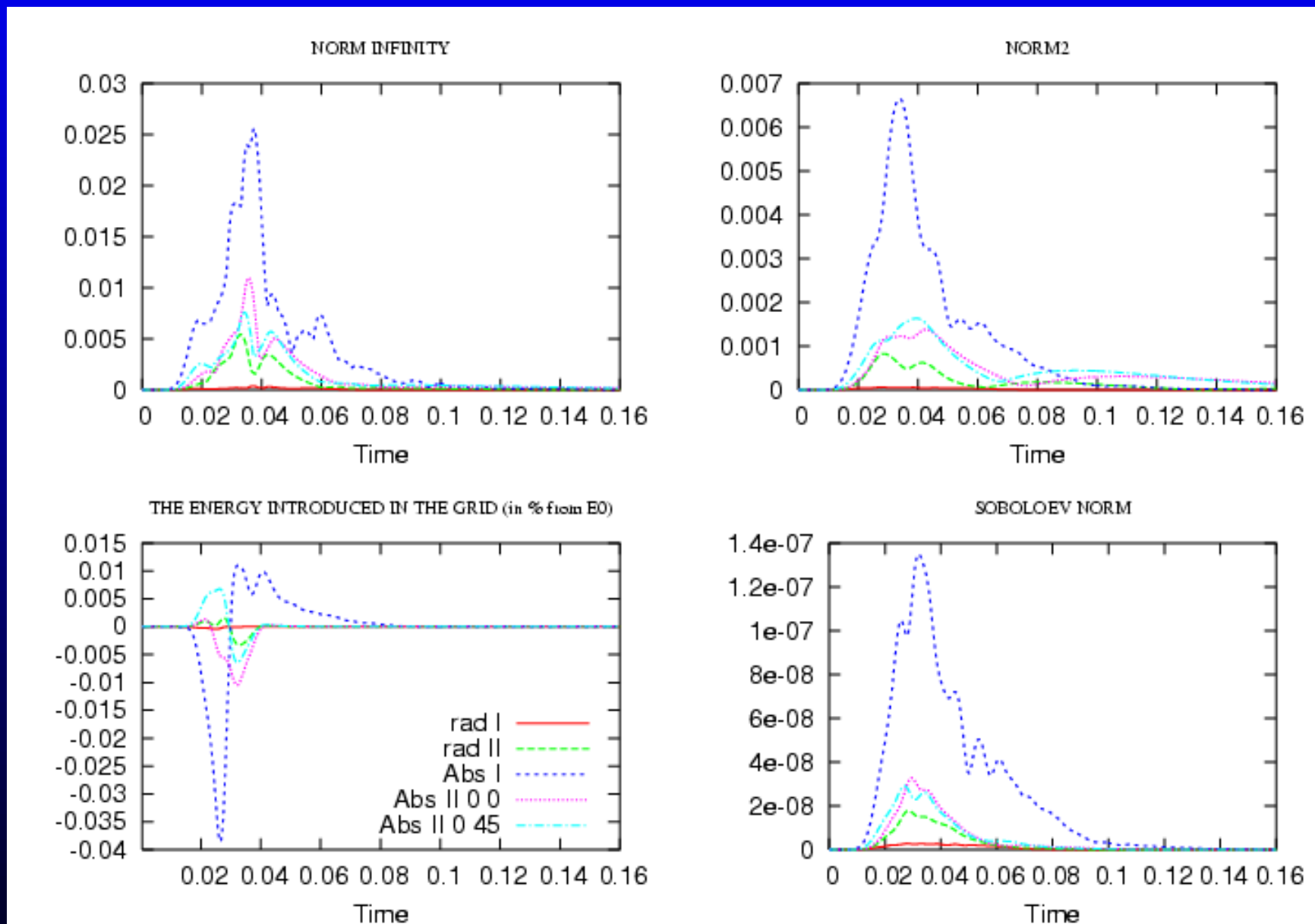


Figure 1: gaussian, centered at origin

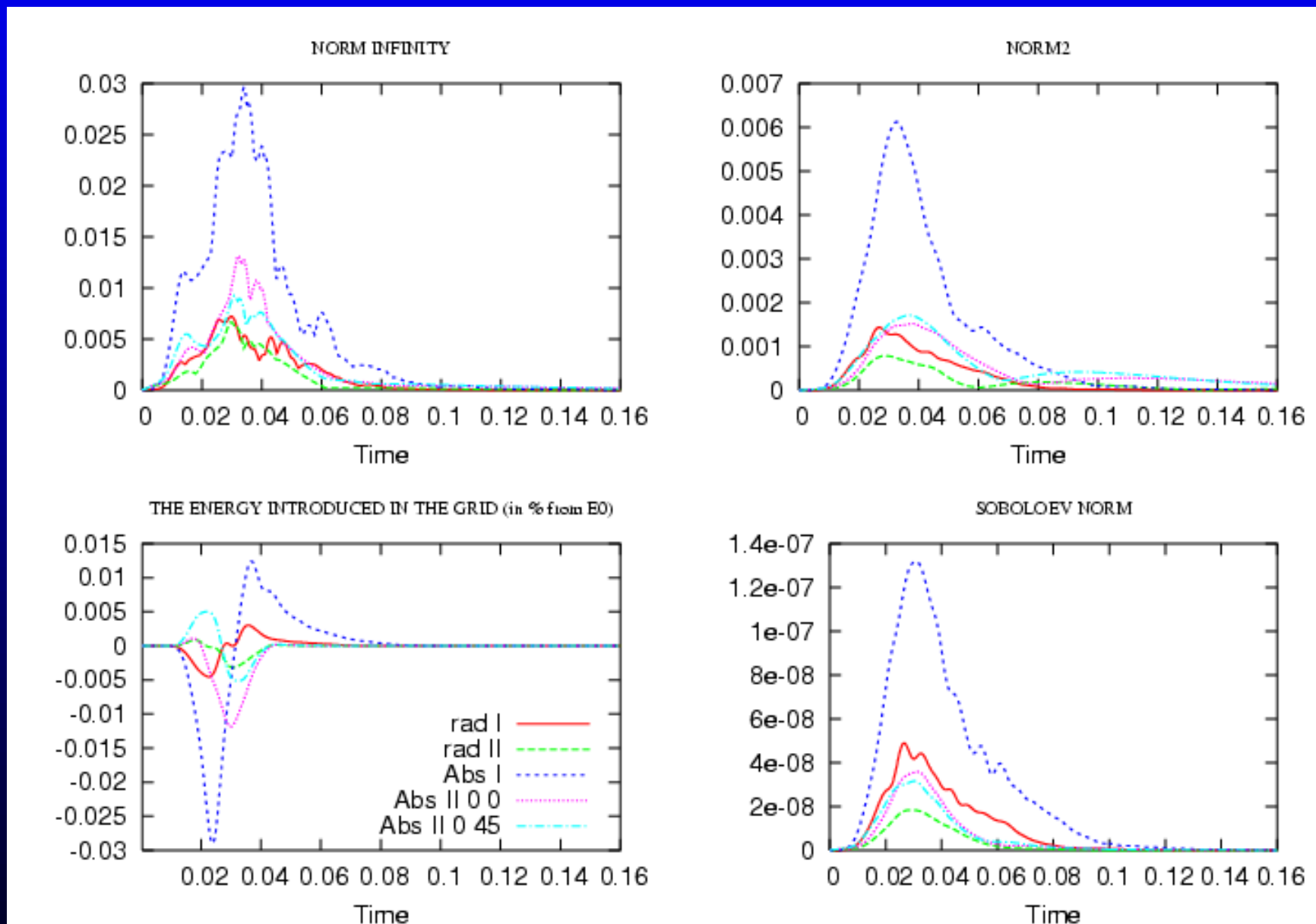


Figure 2: gaussian-offcentered

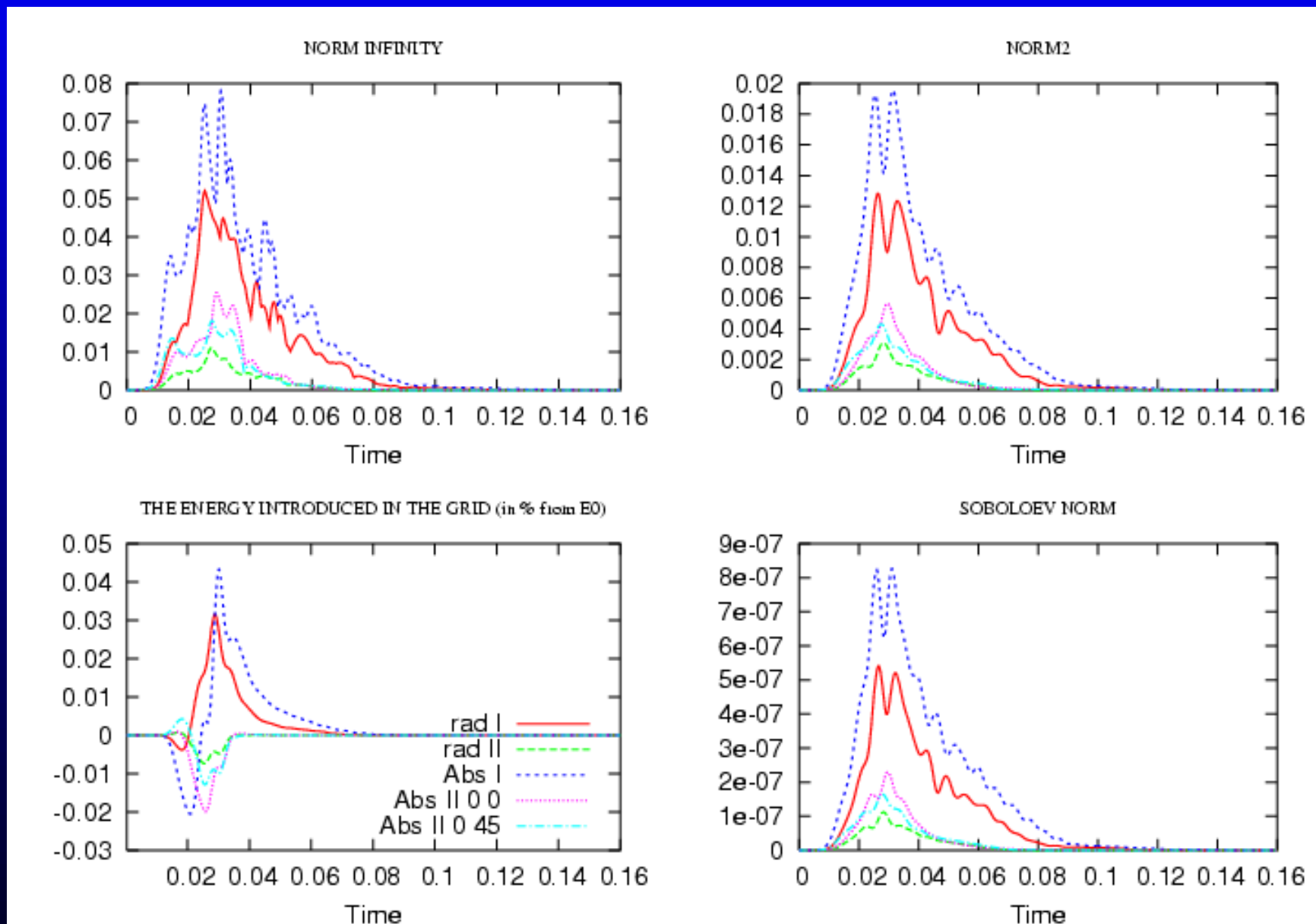


Figure 3: gaussian\*ReY22

## Conclusions

- 'Radiation' BC gave the best results for *spherically symmetric waves centered in origin* – for this situation, the radiation BC is *exact*.
- 'Radiation II' BC gave the best results in all the other cases. (For SSW gives slightly more reflections than 'Radiation')
- The second order absorbing boundary conditions Absorb II (at various angles) gave the best results after Radiation II for *nonspherically symmetric initial data*
- For ID containing  $Y_{lm}$  modes, apparently the 2nd order absorbing BC at 0 and  $45^\circ$  shows better behaviour than the same order BC absorbing only at normal incidence;
- The 1st order absorbing boundary conditions gave the biggest reflections in all the cases

## Outlook

The approach of Engquist/Majda could be applied to to general quasi-linear systems in the following way:

## Outlook

The approach of Engquist/Majda could be applied to general quasi-linear systems in the following way:

*At each evolution step, when the boundary is to be applied . . .*

1. Freeze the coefficients of the system near the boundary. (For instance for quasi-linear systems like Einsteins's field equations this can be done by substituting the data into the coefficients of the derivatives.)

## Outlook

The approach of Engquist/Majda could be applied to to general quasi-linear systems in the following way:

*At each evolution step, when the boundary is to be applied . . .*

1. Freeze the coefficients of the system near the boundary. (For instance for quasi-linear systems like Einsteins's field equations this can be done by substituting the data into the coefficients of the derivatives.)
2. Compute the 'modes' of the resulting constant coefficients linear system and separate them into 'ingoing' and 'outgoing' modes.

## Outlook

The approach of Engquist/Majda could be applied to to general quasi-linear systems in the following way:

*At each evolution step, when the boundary is to be applied . . .*

1. Freeze the coefficients of the system near the boundary. (For instance for quasi-linear systems like Einsteins's field equations this can be done by substituting the data into the coefficients of the derivatives.)
2. Compute the 'modes' of the resulting constant coefficients linear system and separate them into 'ingoing' and 'outgoing' modes.
3. Find a local differential expression that does not contain any mode parameters and *annihilates* outgoing modes of a preferred (e.g., normal) angle of incidence. Adopt as outgoing boundary condition the vanishing of that condition at the boundary.

**nothing**

## other ID

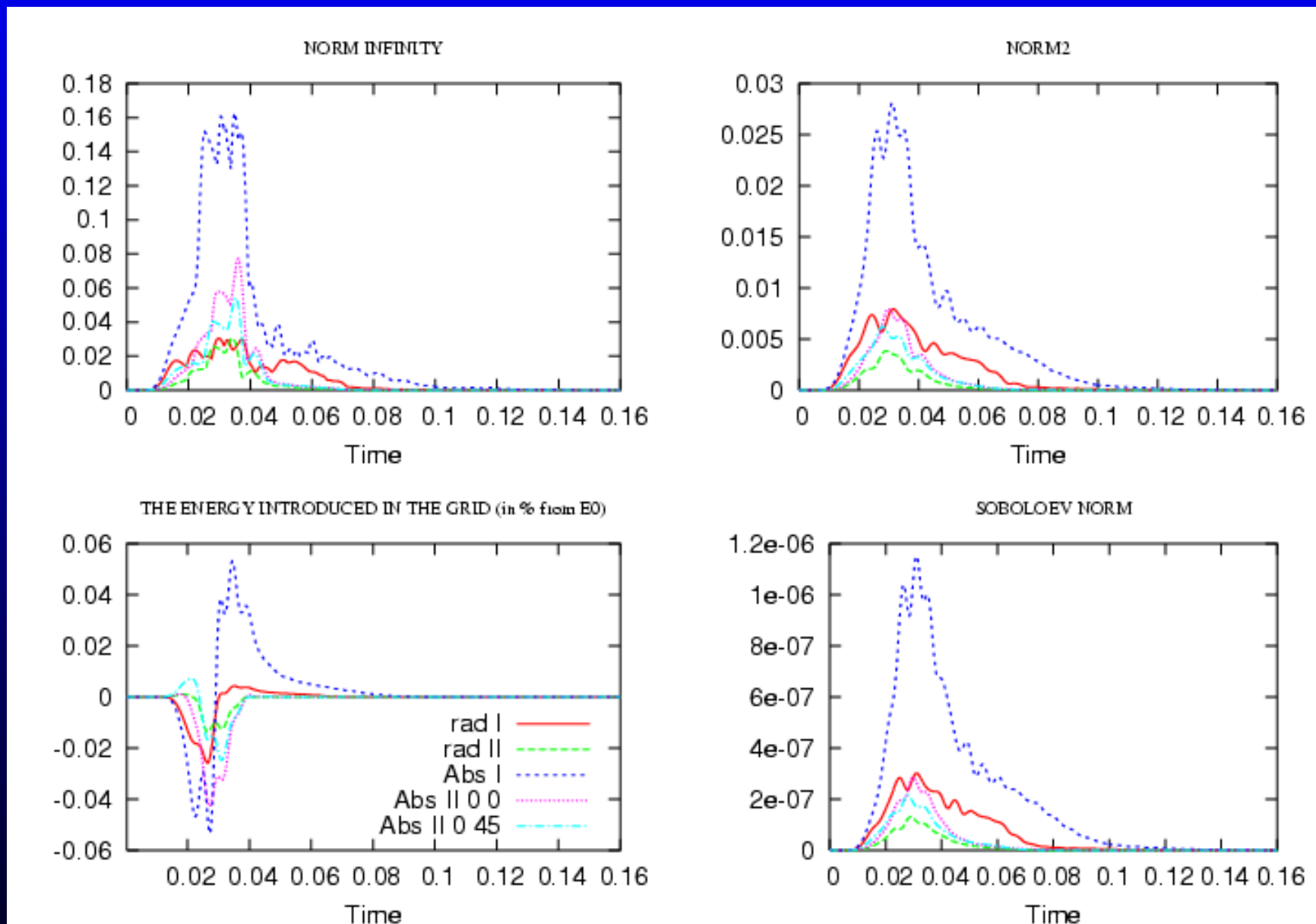


Figure 4: gaussian\*ImY22

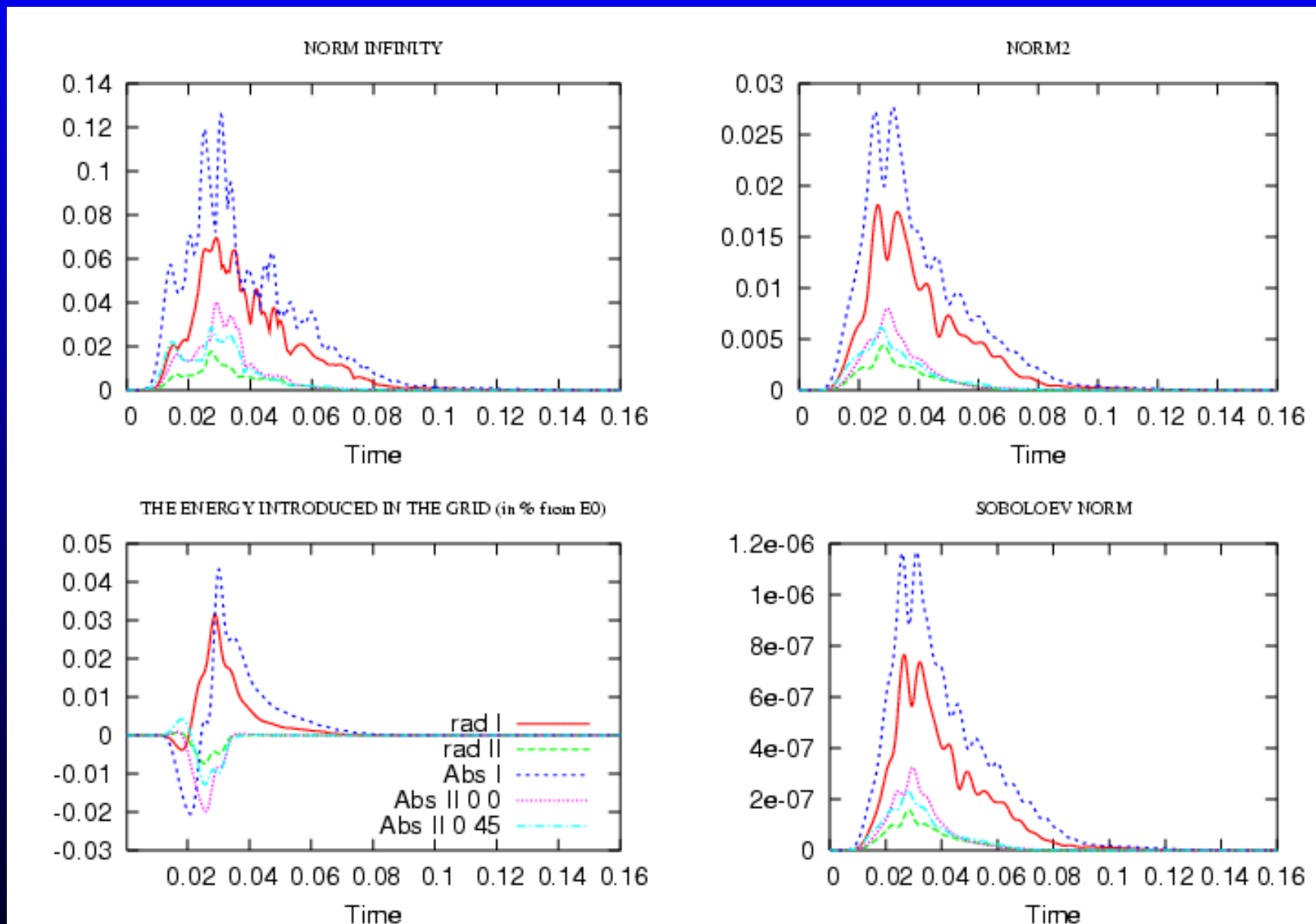


Figure 5: gaussian\*Y20