Constraint-preserving boundary conditions

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Outline

- Dynamics off the constraint surface
- Constraint-preserving boundary conditions
- Detecting ill-posed boundary conditions

Dynamics off the constraint surface

Consider a formulation of Einstein's vacuum equations with a live gauge condition for the lapse (and zero shift):

$$\partial_t N = -NF(N, K, x^\mu), \qquad \frac{\partial F}{\partial K} > 0.$$

This can be incorporated into a first order symmetric hyperbolic formulation (OS, Tiglio). One has evolution equations for the the lapse (N), the three-metric (h_{ij}) and the additional variables $A_i = \partial_i(\log N)$, $d_{kij} = \partial_k h_{ij}$. Symmetric hyperbolicity (and maximal dissipative boundary conditions) guarantee local well posedness which is a necessary condition for the construction of a stable discretization.

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Constraints: Hamiltonian, momentum, $A_i - \partial_i(\log N) = 0$, $d_{kij} - \partial_k h_{ij} = 0$.

Dynamics off the constraint surface

Question:

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We analyze this question by looking at simple solutions of the form

$$
\begin{aligned}
&N = N(t,x), && \\
&h_{xx} = h(t,x), && h_{yy} = h_{zz} = 1, \\
&K_{xx} = k(t,x), && \\
&d_{xxx} = d(t,x), && A_x = A(t,x),
\end{aligned}
$$

all other variables zero.

Dynamics off the constraint surface

Plugging this ansatz into the evolution equations, we obtain a coupled nonlinear system for the variables N, h, k, d, A. All the constraints are identically satisfied except for the two constraints

$$C^{(d)} \equiv d - \partial_x h = 0, \qquad C^{(A)} \equiv A - \partial_x(\log N) = 0.$$

Their propagation is described by the following system of ordinary differential equations

$$\begin{aligned} \partial_t C^{(d)} &= -2NkC^{(A)}, \\ \partial_t C^{(A)} &= -N\frac{\partial F}{\partial N}C^{(A)} + \frac{\partial F}{\partial K}h^{-2}kC^{(d)}. \end{aligned}$$

Dynamics off the constraint surface

For simplicity, we will focus on the time-harmonic gauge, where $F(N, K, x^\mu) = NK$. One can integrate part of the evolution equations to obtain $g(t,x) = F_0(x)N(t,x)^2$ (densitized lapse). It turns out that one can integrate the evolution system for the constraints. The result is

$$\begin{aligned} C^{(d)} &= f_1(x)N^2 + f_2(x)N, \\ C^{(A)} &= f_3(x)N^{-2} + f_4(x)N^{-1}, \end{aligned}$$

where the functions $f_1, ..., f_4$ are determined by the *initial* violation of the constraints.

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Observation: The constraint variables $(C^{(d)}, C^{(A)})$ blow up *if and only if* the lapse becomes singular (0 or ∞). This is a nonlinear effect!

Dynamics off the constraint surface

Using this result, one can reduce all the equations to the following nonlinear wave equation

$$\frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u = \frac{\partial}{\partial x}\left[\epsilon(x)\exp(-u)\right],$$

where $u = \log(N)$, and $\epsilon(x)$ describes the *initial* violation of the constraints.

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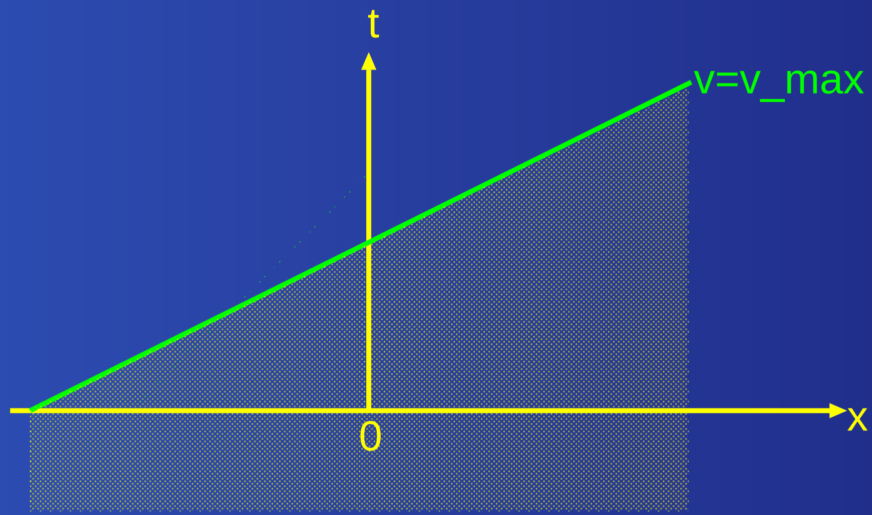
If $\epsilon(x) \equiv 0$, we obtain gauge wave solutions.
What happens if $\epsilon \neq 0$?

Dynamics off the constraint surface

Suppose $\epsilon(x) \equiv 1$. Look for progressive wave solutions (Balabane):

$$u(t,x) = H\left(ct - x\right), \qquad c > 1.$$

Such a solution is given by $H(v) = \log\left[b\, e^{-av} - 1\right] - \log(a)$, where a and b are integration constants. This solution is defined for $v < \log(b)/a$ and blows up as $v \to v_{max} = \log(b)/a$.



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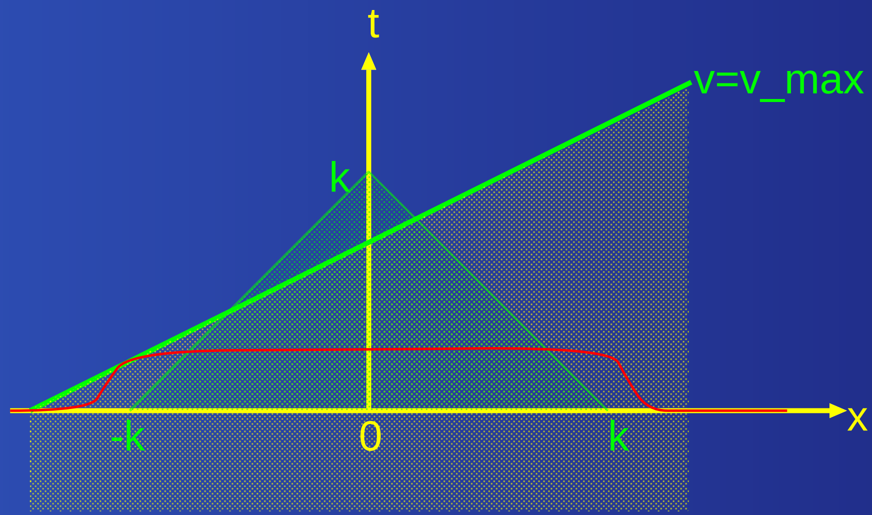


Dynamics off the constraint surface

Therefore, there are solutions with smooth initial data that blow up in finite time. The initial data can be arbitrarily small.

Of course, the crash time becomes larger as the initial data becomes smaller or the initial violation of the constraints becomes smaller.

Constraint-preserving b.c.

Evolution system described by a quasi-linear hyperbolic system:

$$\partial_t u = A^x(u)\partial_x u + A^y(u)\partial_y u + A^z(u)\partial_z u + B(u),$$

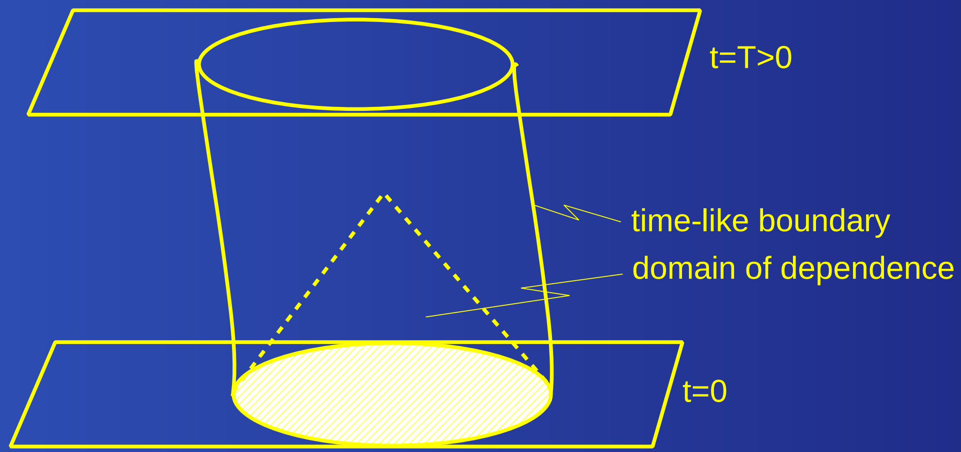
where the matrices A^x, A^y, A^z are symmetric.
Constraints: $C \equiv L^x(u)\partial_x u + L^y(u)\partial_y u + L^z(u)\partial_z u + M(u) = 0$.
Evolution system for the constraint variables (C):

$$\partial_t C = \tilde{A}^x(u)\partial_x C + \tilde{A}^y(u)\partial_y C + \tilde{A}^z(u)\partial_z C + \tilde{B}(u)C,$$

which has the trivial solution $C = 0$. We have to show that this solution is the *unique* solution with trivial initial data (constraint propagation). Furthermore, small initial violations of the constraints should lead to small violations at later times.

Constraint-preserving b.c.

In the absence of boundaries, this follows immediately if the evolution system for the constraint variables is well posed. In the presence of time-like boundaries, the situation is more complicated. One has to make sure that no constraint-violating modes enter from the boundary!



Constraint-preserving b.c.

Strategy: At the boundary, decompose the constraint variables C into in- and outgoing fields (C_{in}, C_{out}). Specify homogeneous *maximal dissipative boundary conditions*:

$$C_{in} = TC_{out} \ ,$$

where T is some coupling matrix (required to be “small enough”).

If the evolution system for the constraint variables is symmetric hyperbolic, an energy estimate guarantees that $C = 0$ is the unique solution with trivial initial data for C.

Reexpress the condition $C_{in} = TC_{out}$ in terms of the main variables.
Yields boundary conditions for the main variables and their derivatives!
These are not directly in maximal dissipative form. Problem!

Constraint-preserving b.c.

Well posed constraint-preserving boundary conditions:

- Friedrich, Nagy: Weyl formulation with tetrad fields, *Comm. Math. Phys.* **201**, 619 (1999).
- Szilágyi, Winicour: formulation with the full harmonic gauge, homogeneous boundary conditions (or small deviations thereof), *Phys. Rev. D* **68**, 041501 (2003).
- Calabrese, Pullin, Reula, OS, Tiglio: Einstein-Christoffel formulation in the weak field limit. *Comm. Math. Phys* **240**, 377 (2003).

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Except for the formulation by Friedrich and Nagy, the situation is unsatisfactory because it fixes a coupling between the “physical” in- and outgoing modes (Dirichlet or Neumann).

Detecting ill-posed b.c.

Calabrese, OS: Necessary conditions for well posedness, *J. Math. Phys* **44**, 3888 (2003).
Consider linear symmetric hyperbolic system with constant coefficients

$$\partial_t u = A^x \partial_x u + A^y \partial_y u + A^z \partial_z u, \quad t > 0, x > 0,$$

with boundary conditions at $x = 0$ of the form

$$M(\partial_t, \partial_y, \partial_z) u = g(t, y, z).$$

(Obtained from the evolution equations and constraint-preserving boundary conditions by freezing the coefficients.)
Look for solutions of the form $u(t, x, y, z) = e^{st + i(w_y y + w_z z)} f(x)$, where $Re(s) > 0$, w_y, w_z real.

Detecting ill-posed b.c.

Test: If $g = 0$ there should be no such solutions. Otherwise the system is ill posed: Because if there is such a solution for some s, $Re(s) > 0$, then there is also a solution for αs, $\alpha > 0$ and for each fixed t

$$|u_\alpha(t,x,y,z)|/|u_\alpha(0,x,y,z)| = e^{\alpha Re(s) t} \to \infty.$$

Introducing the ansatz $u(t,x,y,z) = e^{st+i(w_y y + w_z z)} f(x)$ into the evolution and boundary equations gives

$$\partial_x f = (A^x)^{-1}\left(s - iA^y w_y - iA^z w_z\right) f, \qquad M(s, iw_y, iw_z) f(x=0) = 0.$$

Solution has the form $f(x) = P e^{G_- x} \sigma_-$, $Re(G_-) < 0$ with $MP\sigma_- = 0$. Therefore, one has to verify the determinant condition

$$\det(MP)(s, w_y, w_z) \neq 0, \qquad Re(s) > 0.$$

Detecting ill-posed b.c.

Applications: More general Constraint-preserving boundary conditions for the linearized Einstein-Christoffel system. We take a family of symmetric hyperbolic formulations, parametrized by η.
(The formulation by Anderson & York corresponds to $\eta = 4$.)
The evolution system for the constraint variables is symmetric hyperbolic for $0 < \eta < 2$ and strongly hyperbolic otherwise (well posed initial-value formulation!)

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- Sommerfeld-like conditions for $0 < \eta < 2$: passes test.
- But the test fails for $\eta < 0$ or $\eta > 8/3$.
  Ill posed modes are shown to be constraint-violating.

Consequence: Strong hyperbolicity and maximal dissipative boundary conditions does not imply a well posed initial-boundary value problem!